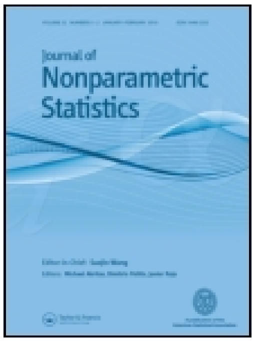

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DELTA METHOD FOR LONG-RANGE DEPENDENT OBSERVATIONS

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It is shown that the delta method is applicable in case of long-range dependent observations i.e. the asymptotic distribution of centered and normalized value of compactly differentiable statistical functional at empirical distribution function coincides with the asymptotic distribution of the linear term in its formal Taylor expansion. As a simple corollary of the result, the asymptotic laws for L-estimates are derived. Also the form of asymptotic laws for M-estimates established by Beran (1991) follows from the main result. The asymptotic law of L-estimates for long-range dependent normal observations does not depend on a specific form of L-estimate. The same conclusion is valid for every compactly differentiable Fisher consistent estimate in the normal location model. Finally a simple sufficient condition for long-range dependence in the subordinated Gaussian model is provided.

KEYWORDS: Long-range dependence, statistical functional, compact differentiability, delta method.

1. INTRODUCTION

Let $Z_1, Z_2, \ldots$ be a stationary Gaussian process with mean $E(Z_1) = 0$, variance $E(Z_1^2) = 1$ and covariance function

$$r(k) = E(Z_1 Z_{k+1}) = k^{-\alpha} L(k), \quad k = 1, 2, \ldots, \tag{1.1}$$

where $0 < \alpha < 1$ and $L(\cdot)$ is a function on $[1, \infty)$ that is slowly varying at infinity and is positive in some neighborhood of infinity. Such a sequency $\{Z_j\}_{j=1}^{\infty}$ is said to exhibit long-range dependent behaviour. Let $G(\cdot)$ be an arbitrary Borel measurable function on the real line $\mathbb{R}$ and consider the process $X_i = G(Z_i)$, $i = 1, 2 \ldots$ with marginal distribution function $F(x) = P(X_1 \le x)$. Then the derived process X_j, $j = 1, 2, \ldots$ will also exhibit a long-range dependent behaviour at least as α is small enough as determined by the Hermite rank m^* of the function $G(\cdot)$ defined below (cf. Taqqu (1975), Lemma 3.1). We refer to Beran (1992) for the general discussion of long-range dependence and motivating examples. Let $F_n(x) = n^{-1} \sum_{i=1}^{n} I\{X_i \le x\}$ be the sample distribution function, where $I\{A\}$ denotes the indicator function of an event A. Observe that

$$F_n(x) - F(x) = \frac{1}{n} \sum_{i=1}^{n} G_x(Z_i),$$

where $G_x(\cdot) = I\{G(\cdot) \le x\} - F(x)$. Let $H_q(s) = (-1)^q e^{s^2/2} (d^q/ds^q) e^{-s^2/2}$, $s \in \mathbb{R}$ be the q^{th} Hermite polynomial, $q = 1, 2 \ldots$ and $H_0(\cdot) = 1$. Hermite polynomials form an orthonormal, complete system in the space $\mathscr{L}^2(\mathbb{R}, \phi)$ with the weight function ϕ being the standard normal density. Consider the Fourier-Hermite expansion of $G_x(\cdot)$ by $\{H_q(\cdot)\}_{q=0}^{\infty}$

$$G_x(\cdot) = \sum_{q=m(x)}^{\infty} \frac{J_q(x)}{q!} H_q(\cdot) \tag{1.2}$$

for fixed $x \in \mathbb{R}$. The coefficients $J_q(x)$ in (1.2) are equal to $E(H_q(Z_1)G_x(Z_1))$, $q = 0, 1 \ldots$ and $m(x)$ is the smallest index $q \in \mathbb{N}$ for which $J_q(x) \neq 0$. Observe that $m(x) > 0$ since $EG_x(Z_1) = 0$ for $x \in \mathbb{R}$. The Hermite rank of the class of functions $\{G_x(\cdot) : x \in \mathbb{R}\}$ is defined as $m = \inf\{m(x) : x \in \mathbb{R}\}$. It is assumed throughout the note that

$$0 < \alpha < \tfrac{1}{m}. \tag{1.3}$$

It follows from Lemma 3.1 in Taqqu (1975) that the variance $d_{mn}^2 = Var(\sum_{i=1}^{n} H_m(Z_i))$ is such that

$$\frac{n}{d_{mn}} \sim C(m, \alpha) \frac{n^{m\alpha/2}}{L^{m/2}(n)}, \text{ where } C(m, \alpha) = \sqrt{\frac{(1 - m\alpha)(2 - m\alpha)}{2m!}}, \tag{1.4}$$

as $n \to \infty$. Let $D[-\infty, \infty]$ denote the non-separable metric space of all real functions defined on $[-\infty, \infty]$, which are right-continuous and have left-hand limits with the metric pertaining to the supremum norm $\|\cdot\|_\infty$. Dehling and Taqqu (1989) proved that

$$\frac{n}{d_{mn}} \{F_n(\cdot) - F(\cdot)\} \to T_m(\cdot) = \frac{J_m(\cdot)}{m!} Y_m \text{ in } D[-\infty, +\infty],$$

is distribution with respect to the σ-algebra generated by the family of open balls. Y_m is some non-degenerate random variable which is normal for $m = 1$ but is not normally distributed for $m = 2, 3, \ldots$ Observe that in view of (1.3) and (1.4) the normalizing factor for the weak convergence is $o(n^{1/2})$ in constrast with the case of i.i.d. observations.

Let now X be a normed vector space and $T: X \to \mathbb{R}$ a statistical functional. Consider the problem of estimating $T(F)$ by means of $T(F_n)$. The following definition is relevant.

DEFINITION T is boundedly (compactly) differentiable at $x \in X$ if there exists a linear continuous mapping $dT(x) : X \to \mathbb{R}$ such that

$$\lim_{t \to 0} \sup_{y \in B} \frac{\|T(x + ty) - T(x) - \mathrm{d}T(x)ty\|}{t} \to 0,$$

where B denotes arbitrary bounded (compact) subset of X.

The usefulness of the concept of compact derivative is apparent in view of the following lemma (see e.g. Rieder (1994), Theorem 1.3.3). The lemma carefully avoids possible nonmeasurability of the set $\{a_n Z_n \in K_n^\delta\}$ occuring e.g. when $a_n Z_n$ is an empirical process considered as the mapping into $(D[-\infty, \infty], \|\cdot\|_\infty)$ with the σ-field generated by the family of open ballls. Other approaches providing justification for the delta method exist (see e.g. Gill, 1989; Esty *et al.*, 1985) but this seems to be the most direct one. P_* denotes the inner probability pertaining to P and A^δ is the set of points x such that distance from x to A does not exceed δ.

LEMMA Suppose that T is compactly differentiable at some $x \in X$. Consider a sequence of functions Z_n from probability spaces (Ω, A_n, P_n) to X. Assume that for every $\varepsilon > 0$ there exist a compact subset K of X and sequences $\delta_n \to 0$ and $a_n \to \infty$ such that

$$\liminf_{n \to \infty} P_{n*}(a_n Z_n \in K_n^{\delta}) \geq 1 - \varepsilon.$$

Then

$$a_n(T(x + Z_n) - T(x)) = dT(x) a_n Z_n + {}^{\circ}P_{n*}(1). \tag{1.5}$$

In the case when X_i are independent and identically distributed random variables the above lemma provides a theoritical justification of the delta method which states that the asymptotic distribution of $n^{1/2}(T(F_n) - T(F))$ is the same as that of $dT(F)(B(F(\cdot))$, where $B(\cdot)$ denotes Brownian bridge. In the note we prove that in analogy to this result in case of long-range dependent sequence $X_i, i = 1, 2, \ldots$, $n\, d_{mn}^{-1}(T(F_n) - T(F))$ has the same limiting law as $Y_m/m!\, dT(F) J_m(\cdot)$. The asymptotic laws for M- and L-estimates in case of long-range dependence are simple consequences of this result. In particular, it turns out that it case of $G(x) = x$ i.e. normal long-range dependent sequence Z_i, the asymptotic distribution of L-estimates does not dependend on a weight function. The same conclusion is valid for every compactly differentiable Fisher consistent statistical functional in the normal location model. This generalizes the result established by Beran (1991) for M-estimates in this model but is in complete contrast with the i.i.d. case. Finally we prove that the condition (1.3) assumed throughout this note is implied by the more easily verifiable condition $m^*\alpha < 1$ where m^* is the Hermite rank of the function $G(\cdot)$. The proofs are postponed until Section 3.

2 RESULTS

Let $\xrightarrow{\mathscr{D}}$ denotes the convergence in distribution of a sequence of real random variables. Then main result is the following

THEOREM Assume that $m\alpha < 1$ and T: $(D[-\infty, \infty], \|\cdot\|_\infty) \to \mathbb{R}$ is a statistical functional compactly differentiable at F. Then

$$\frac{n}{d_{mn}}(T(F_n) - T(F)) = dT(F)\frac{n}{d_{mn}}(F_n(\cdot) - F(\cdot)) + {}^{\circ}P_{n*}(1) \tag{2.1}$$

and

$$\frac{n}{d_{mn}}(T(F_n) - T(F)) \xrightarrow{\mathscr{D}} \frac{Y_m}{m!} dT(F)\, J_m(\cdot). \tag{2.2}$$

Consider L-functional defined by

$$T(F) = \int F^{-1}(s) J(s)\, ds, \tag{2.3}$$

where the weight function $J(\cdot)$ and c.d.f $F(\cdot)$ satisfy the following conditions:

(i) $J \in \mathscr{L}^2[0, 1]$ and $\int J(s)\mathrm{d}s = 1$;
(ii) $J(\cdot)$ is a bounded measurable function supported on $[\alpha, \beta]$, where $0 < \alpha < \beta < 1$;
(iii) the Lebesgue measure of the set $\{y \in \mathbb{R} : J(\cdot)$ is discontinuous at $F(y)\}$ is 0.

COROLLARY 1 Assume that conditions (i)–(iii) hold. Then for T defined in (2.2) we have

$$\frac{n}{d_{mn}}(T(F_n) - T(F)) \xrightarrow{\mathscr{D}} \frac{Y_m}{m!} \int J(F(y))\, J_m(y)\, \mathrm{d}y. \tag{2.4}$$

EXAMPLE 1 The trimmed mean corresponds to $J(t) = I(t \in [\alpha, 1-\alpha])/(1-2\alpha)$, $0 < \alpha < 1/2$. This weight function clearly satisfies (i)–(iii).

Remark 1 Observe that in the case when $G(x) = x$, we have $m = 1$ and $J_1(x) = \int_{-\infty}^{x} s\phi(s)\,\mathrm{d}s = -\phi(s)$. Thus $\int J(F(y))\, J_1(y)\,\mathrm{d}y = -\int J(s)\mathrm{d}s = -1$ so that in case of long-range dependent normal errors the asymptotic law of L-estimate does not depend on its specific form.

Consider now the fixed function $\psi\colon \mathbb{R} \to \mathbb{R}$ and the family $\{\psi(\cdot - \theta)\}$, where θ belongs to a compact set $\Theta \subset \mathbb{R}$ such that for some θ_0 in the interior of Θ $\eta(\theta_0) = 0$, where $\eta(\theta) = \int \psi(x-\theta)\mathrm{d}F(x)$. Provided that $\eta(\cdot)$ is locally homeomorphic at θ_0, there exists a neighborhood V of $\eta(\cdot)$ in $C(\Theta, \mathbb{R})$ and a functional $T\colon V \to \Theta$ such that for every $f \in V, f(T(f)) = 0$ (cf. Rieder (1994), Theorem 1.4.2). Let BV be the space of functions of bounded variation on $\mathbb{R}$. For $G \in D[-\infty, \infty] \cap BV$ put $T_0(G) = \int \psi(x - \cdot)dG(x)$ and assume that $T_0(G)$ is continuous for every G. We define M-functional on $T_0^{-1}(V)$ as $T^*(G) = T \circ T_0(G)$. Consider now the model $Y_i = G(Z_i) + \theta_0, i = 1, 2, \ldots, n$, where $Y_i \sim F$. From now on F_n denotes the e.d.f. based on $(Y_i)_{i=1}^n$. Observe that the Hermite ranks pertaining to the function $G(\cdot)$ and $G(\cdot) + \theta_0$ are the same.

COROLLARY 2 Assume that $\eta(\cdot)$ is locally homeomorphic at $\theta_0, \eta'(\cdot)$ exists at θ_0 and $\eta'(\theta_o) \neq 0, T_0(G)(\cdot)$ is continuous for every $G \in BV \cap D[-\infty, \infty]$ and $T_0\colon BV \cap D[-\infty, \infty] \to (C(\theta, \mathbb{R}), \|\cdot\|_\infty)$ is continuous. Then

$$\frac{n}{d_{mn}}(T^*(F_n) - T^*(F)) = -\frac{1}{d_{mn}} \frac{\sum_{i=1}^n \psi(Y_i - \theta_0)}{\eta'(\theta_0)} + {}^{\circ}P_{n*}(1). \tag{2.5}$$

EXAMPLE 2 Some well known score functions are $\psi(x) = \operatorname{sgn}(x)$ (median), $\psi(x) = x$ (mean) and $\psi(x) = \max(-c, \min(x, c))$ for some constant $c > 0$ (Huber's estimator).

Remark 2 Observe that under regularity conditions on $\eta(\cdot)$ entailing

$$\eta'(\theta_0) = \frac{\mathrm{d}}{\mathrm{d}\theta} \int \psi(x-\theta)\,\mathrm{d}F(x)_{|\theta=\theta_0} = -\int \psi'(x-\theta_0)\,\mathrm{d}F(x),$$

the main term on the right-hand side of (2.5) can be written as

$$\frac{1}{d_{mn}} \frac{\sum_{i=1}^n \psi(X_i)}{E\psi'(X_1)}. \tag{2.6}$$

Consider the case when the rank of $\psi \circ G$ is equal to the rank of G and is equal to 1. Then the asymptotic distribution of (2.6) in view of Taqqu (1975) is equal to the distribution of

$$\frac{\int \psi(G(z)) z \phi(z) \, dz}{\int \psi'(G(z)) \phi(z) \, dz} Z,$$

whereas the asymptotic distribution of the sample mean is $\int G(z) z \phi(z) \, dz Z$, Z being some standard normal random variable. This result was first proved by Beran (1991). In particular if $G(x) = x$ and $\psi(\cdot)$ is an arbitrary monotone antisymmetric function, these asymptotic distributions are equal, analogously as for L-estimates. Observe however that the last conclusion is implied by the following general result. Consider the location model $f_{\theta_0}(\cdot) = \phi(\cdot - \theta_0)$, $\theta_0 \in \mathbb{R}$ and let $F_\theta(\cdot)$ be the distribution function corresponding to $f_\theta(\cdot)$, $\theta \in \mathbb{R}$. A statistical functional T is called Fisher consistent if $T(F_\theta) = \theta$ for every $\theta \in \mathbb{R}$.

COROLLARY 3 Let $T: (D[-\infty, \infty], \|\cdot\|_\infty) \to \mathbb{R}$ be compactly differentiable Fisher consistent statistical functional. Then

$$\frac{n}{d_{mn}} (T(F_n) - \theta_0) \xrightarrow{\mathscr{D}} Z.$$

Observe now that the crucial condition (1.3) of the long-range dependence of the sequence $G_x(Z_i), i = 1, 2, \ldots$ for $x \in \mathbb{R}$ is implied by more easily verifiable condition $m^* \alpha < 1$, when m^* is the Hermite rank of the function $G(\cdot)$ i.e. the smallest $m \in \mathbb{N}$ such that $I_m = E(H_m(Z_1) G(Z_1)) \neq 0$. This follows from the following

PROPOSITION Assume that $G(\cdot) \in \mathscr{L}^2(\mathbb{R}, \phi)$ and $EG(Z_1) = 0$. Then there exist $A \in \mathfrak{B}(\mathbb{R})$ of positive Lebesgue measure such that $J_{m^*}(x) \neq 0$ for $x \in A$.
Observe that the strict inequality $m < m^*$ is possible.

EXAMPLE 3 Consider $0 < a < b < \infty$ and a function

$$W(t) = I_{[0, \infty]}(t) (c I_{\mathbb{R} \setminus (a,b)}(t) - I_{(a,b)}(t)) \quad t \in \mathbb{R},$$

where $c > 0$ is chosen so that

$$I_1 = c \int_{\mathbb{R}^+ \setminus (a,b)} t \phi(t) \, dt - \int_a^b t \phi(t) \, dt) = 0.$$

Put $G(\cdot) := W(\cdot) - EW(Z_1)$. In view of the above equality the Hermite rank of $G(\cdot)$ is not smaller than 2. Suppose now that $-1 < x < 0$ and let $y = x - EW(Z_1)$. Then $G_y(t) = I(W(t) \leq x) - F(y) = I(t \in (a, b)) - F(y)$. Thus $J_1(y) = \int G_y(t) t \phi(t) \, dt = \int_a^b t \phi(t) \, dt > 0$, implying $m(y) = 1$ and $m = 1$.

3 PROOFS

Proof of Theorem Dehling and Taqqu (1989) proved that

$$W_n = \sup_x \left| \frac{n}{d_{mn}} (F_n(x) - F(x)) - \frac{J_m(x)}{d_{mn} m!} \sum_{i=1}^n H_m(X_i) \right| \to 0 \quad \text{a.s.} \tag{3.1}$$

Fix $\varepsilon > 0$ and $\delta > 0$. Put $A_{n,\delta} = \{W_m \le \delta \text{ for all } m \ge n\}$ for $n \in \mathbb{N}$. Since $A_{n,\delta} \subseteq A_{n+1,\delta}$ and in view of (3.1) $\bigcup_{n=1}^{\infty} A_{n,\delta} = \Omega$ it follows that for $n \ge n(\delta)$

$$P(A_{n,\delta}) \ge 1 - \varepsilon. \tag{3.2}$$

Moreover

$$d_{mn}^{-1} \sum_{i=1}^{n} H_m(X_i) \xrightarrow{\mathscr{D}} Y_m \tag{3.3}$$

in $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ (cf. Dobrushin, Major (1979) and Taqqu (1979)). Thus the tightness of the above sequence implies that there exists $0 < M_\varepsilon < \infty$ such that for every n

$$P(d_{mn}^{-1} \textstyle\sum_{i=1}^{n} H_m(X_i) \in [-M_\varepsilon, M_\varepsilon]) > 1 - \varepsilon. \tag{3.4}$$

The set $K = \{(J_m(\cdot)/m!)\, a : a \in [-M_\varepsilon, M_\varepsilon]\}$ is compact in $(D[-\infty, \infty], \|\cdot\|_\infty)$ and from (3.2) and (3.4) it follows that for $n \ge n(\delta)$

$$P_{n*}\left(\frac{n}{d_{mn}}(F_n(\cdot) - F(\cdot)) \in K^\delta\right) \ge 1 - 2\varepsilon. \tag{3.5}$$

Without loss of generality we assume that $n(\delta)$ is monotone in δ. Define $\delta_n = 2^{-i}$ for $n(2^{-i}) \le n < n(2^{-i-1})$, where $i \in \mathbb{N}$. Then (3.5) implies that

$$\liminf_{n \to \infty} P_{n*}\left(\frac{n}{d_{mn}}(F_n(\cdot) - F(\cdot)) \in K^{\delta_n}\right) \ge 1 - 2\varepsilon.$$

Thus the assumptions of the Theorem are satisfied with $Z_n = F_n(\cdot) - F(\cdot)$ and $a_n = n/d_{mn}$. The first assertion of the Theorem is proved. In order to prove the second assertion observe that continuity of $dT(F)$ and (3.1) implies

$$\left| \frac{n}{d_{mn}} dT(F)(F_n(\cdot) - F(\cdot)) - \left\{ \frac{1}{d_{mn}} \sum_{i=1}^{n} H_m(Z_i) \right\} dT(F) \frac{J_m(\cdot)}{m!} \right| \to 0 \quad \text{a.s.}$$

The weak convergence in (2.2) follows from the above convergence and (3.3) in conjunction with (2.1). ■

Proof of Corollary 1 Observe that (1.5) holds when T has bounded derivative along the subspace $Z \subset X$ (c.f. Rieder, Definition 1.3.1) provided that Z_n is a function into Z and $Z_n \to 0$ in probability P_{n*}. Put $\mathscr{H} = \{r(G - F) : r \in \mathbb{R}, G \in \mathscr{F}\}$, where $\mathscr{F}$ is the space of cummulative distributions functions and $Z_n = F_n - F$. Then $Z_n \to 0$ in probability in view of Dehling and Taqqu (1989) and Thas a bounded derivative at F along $\mathscr{H}$ and $dT(F)(G) = -\int J(F(s))\, G(y)\, dy$ provided the assumptions (i)–(iii) hold in view of Theorem 1.6.8 in Rieder (1994). ■

Proof of Corollary 2 The proof follows from the fact that T is compactly differentiable with the derivative

$$dT(F)(f) = -\frac{f(\theta_0)}{\eta'(\theta_0)},$$

(c.f. Rieder (1994), Theorem 1.4.2) and T_0 is boundedly differentiable as the continuous linear functional. Thus $T \circ T_0$ is compactly differentiable and the Corollary follows from the first assertion of the Theorem and the remark that

$$dT^*(F)(F_n - F) = -\frac{\int\psi(x-\theta_0)(F_n - F)(\mathrm{d}x)}{\eta'(\theta_0)} = -\frac{\int\psi(x-\theta_0)F_n(\mathrm{d}x)}{\eta'(\theta_0)}. \qquad \blacksquare$$

Proof of Corollary 3 Observe that the location model corresponds to $G(x) = x + \theta_0$, $x \in \mathbb{R}$. In view of Taqqu (1975), Theorem implies that

$$\frac{n}{d_{mn}}(T(F_n) - T(F_{\theta_0})) \xrightarrow{\mathscr{D}} Z\,dT(F_{\theta_0})\,J_1(\cdot),$$

where $J_1(x) = \int_{-\infty}^{x-\theta_0} s\phi(s)\,\mathrm{d}s = -\phi(x-\theta_0)$ and Z is some standard normal random variable. We prove that $dT(F_{\theta_0})J_1(\cdot) = 1$. Let $h_{\theta_0}(\cdot) := J_1(\cdot), x := F_{\theta_0}(\cdot)$ and $h_\theta(\cdot) := (F(\cdot - \theta) - F(\cdot - \theta_0))/(\theta - \theta_0)$ for $\theta \neq \theta_0$. Observe that in view of uniform continuity of $\phi(\cdot)$, mean value theorem implies that $\|h_\theta - h_{\theta_0}\|_\infty \to 0$ when $\theta \to \theta_0$. Using Proposition 1.3.2 in Reider (1994) with $X := (D[-\infty, \infty], \|\cdot\|_\infty)$ and $K = \{h_{\theta_0}\}$ we have

$$\begin{aligned} T(F_\theta) &= T(x + (\theta - \theta_0)h_\theta) = T(x) + (\theta - \theta_0)\,dT(F_{\theta_0})h_\theta + o(|\theta - \theta_0|\,\|h_\theta\|_\infty) \\ &= T(x) + (\theta - \theta_0)\,dT(F_{\theta_0})h_{\theta_0} + o(|\theta - \theta_0|), \end{aligned}$$

where the last equality follows from the continuity of $dT(F_{\theta_0})$ and the fact that the family $\{h_\theta\}_{\theta\in\mathbb{R}}$ is uniformly bounded. Taking derivatives with respect to θ yields

$$1 = \frac{\mathrm{d}}{\mathrm{d}\theta}T(F_\theta)_{|\theta=\theta_0} = dT(F_{\theta_0})h_{\theta_0}. \qquad \blacksquare$$

Proof of Proposition Let $G^a(t) = \min(a, G(t))$ for $a > 0$ and $I_l = \int G(t)H_l(t)\phi(t)\,\mathrm{d}t$, $l \in \mathbb{N}$. Observe that for $l \in \mathbb{N}\setminus\{0\}$

$$\begin{aligned} \int_{-\infty}^{a} J_l(x)\,\mathrm{d}x &= \int_{-\infty}^{a}\int_{G(t)\le x} H_l(t)\phi(t)\,\mathrm{d}t\,\mathrm{d}x = \int_{-\infty}^{a}\int_{G_a(t)\le x} H_l(t)\phi(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}(a - G_a(t))\,H_l(t)\,\phi(t)\,\mathrm{d}t = -\int G_a(t)\,H_l(t)\,\phi(t)\,\mathrm{d}t. \end{aligned}$$

Note that $\int|G(t)\,H_l(t)\phi(t)|\,\mathrm{d}t \le (EG^2(Z))^{1/2}(l!)^{1/2} < \infty$ thus from Lebesgue dominated convergence theorem $\int G_a(t)\,H_l(t)\phi(t)\,\mathrm{d}t \to I_l$, when $a \to \infty$. Put $l = m^* > 0$. Then $\int J_{m^*}(x)\,\mathrm{d}x \neq 0$ and the conclusion follows from properties of Lebesgue integral. $\blacksquare$

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