Stockholm University
Department of Statistics
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Econometrics I

WRITTEN EXAMINATION

Thursday April 27, 2017, 10 am - 3 pm

Allowed tools: Pocket calculator

Passing rate: 50% of overall total, which is 100 points. For detailed grading criteria, see the course description.

The exam will be handed back on Monday May 15 at 3 pm in room B705.

For the maximum number of points on each problem detailed and clear solutions are required.

If not indicated otherwise, the disturbance terms u_i in the models are supposed to fulfill the usual requirements of normality, homoscedasticity and independence.

You may answer in Swedish.

1. (25p) One wants to study the effective life time Y (in minutes)) of a cutting tool on a lathe, with speed X (in rounds per minute), where we have access to two types of tools (A and B).

We assume the following model:

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 D_i + u_i.$$

where

$$D_i = \begin{cases} 1 & \text{if type B} \\ 0 & \text{if type A} \end{cases}$$

For data where we have used 10 tools of type A and 10 tools of type B ($n = 20$), we got the following results:

ANOVA

Source	Sum of Squares	df
Regression (SSE)	1418.034	2
Residuals (SSR)	157.055	17
Total (SST)	1575.089	19

Coefficients Table

Variable	Coefficient	SE(β_i)
Constant	36.986	
X	-0.027	0.005
D	15.004	1.36

(a) Compute both coefficients of (multiple) determination (nonadjusted and adjusted).

(b) Investigate by a test if at least one of the explanatory variables should be included in the model. Use proper notation and state clearly the null and alternative hypotheses. Use significance level 1%.

(c) How do you interpret the parameter β_3 in terms of expectation?

(d) Compute a 95% confidence interval for β_3.

(e) An alternative model is

$$Y_i = \beta_1' + \beta_2' X_i + \beta_3' D_i + \beta_4' X_i D_i + u_i'.$$

Describe the difference between the two models, that is, in what specific way is the second model more flexible ("richer")?

(f) The alternative second model was also used for estimation with the same data. Results:

ANOVA

Source	Sum of Squares	df
Regression (SSE)	1434.112	3
Residuals (SSR)	140.977	16
Total (SST)	1575.089	19

Coefficients Table

Variable	Coefficient	SE($\hat{\beta}_i$)
Constant	32.775	
X	-0.021	0.0061
D	23.971	6.769
XD	-0.012	0.088

Is the second model siginficantly better than the first model? Use significance level 5% with a suitable test.

2. (25p) The Cobb-Douglas production function is

$$Q = \beta_1 L^{\beta_2} K^{\beta_3}. \tag{1}$$

where L = labour input and K = capital stock.

Now, we want to use $Y = ln(Q)$ as dependent variable in a linear regression model.

(a) Write down a linear regression model based on (2) using Y as defined above.

(b) Now let $\beta_3 = 1 - \beta_2$. Rewrite the linear model according to this.

(c) Without doing any computation (since we do not have the numerical results, except that we assume that $n = 33$), describe which test you would use for comparing the model in (a) with the model in (b).

In doing so, describe the null hypothesis, the test statistic with its parameters and for which values of the test statistic you should reject the null hypothesis.

(d) Mention one potential advantage, from a regression modelling point of view, of using the model in (b) instead of the model in (a).

(e) Using data from the years 1899 to 1922, we got that using the model in (a), the residual-based estimated autocorrelation coefficient $\hat{\rho} = 0.6$. Can we from this draw the conclusion that we have evidence of positive autocorrelation at (approximate) significance level 1%?

3. (20p) In order to investigate what proportion (on average) of the income a person spends on the rent of a flat in New York, data were collected from 108 single households. A simple linear regression model was estimated using rent as dependent variable (Y) and income as independent variable (X). One suspects problems with heteroscedasticity. White's test was performed with the following results:

White's Auxiliary Regression

Included Observations: 108

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	−14657900	9288994	−1.577986	0.1176
INCOME	1200.579	495.1663	2.424598	0.0170
INCOME^2	−0.010007	0.005355	−1.868714	0.0644

R-squared	0.082134	Mean dependent var	10515952
Adjusted R-squared	0.064651	S.D. dependent var	29847739
S.E. of regression	28866783	Akaike info criterion	37.22167
Sum squared resid	8.75E + 16	Schwarz criterion	37.29617
Log likelihood	−2006.970	F-statistic	4.697874
Durbin–Watson stat	1.864571	Prob(F-statistic)	0.011115

(a) Write down the auxiliary regression model used in this situation for White's test with proper notation.

(b) Show that we can reject the null hypothesis of homoscedasticity at significance level 5%

(c) It turns out that we can model the variance of the disturbance variable as $V(u_i) = \sigma^2 X_i$. How can we take this into account to obtain a homoscedastic model?

4. (12p) The " true" model given specific data is supposed to be

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

For some reason though, the following model is used for estimation of the β-parameters:

$$(Y_i - c_1) = \beta_1 + \beta_2(X_i - c_2) + u_i.$$

where c_1 and c_2 are given (known) constants.

(a) Derive expressions for the OLS-estimators of β_1 and β_2 using the second "wrong" model.

(b) Are both estimators from (a) unbiased?

5. (18p) True or false statements? Short motivation/comment also needed.

(a) If X_i in the model $Y_i = \beta_1 + \beta_2 X_i + u_i$ is divided by 2, both estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ are changed.

(b) If we in the model $Y_i = \beta_1 + \beta_2 X_i + u_i$ have a measurement error in Y_i, then $Cov(X_i, u_i) \neq 0$.

(c) The Durbin-Watson test cannot be used if the disturbance variables are autocorrelated.

(d) For two random variables, X_i and Y_i, we always have that $-1 \leq Cov(X_i, Y_i) \leq 1$.

(e) Heteroscedasticity occurs when the disturbance term in a regression model is correlated with one of the explanatory variables.

(f) In the Runs test, the number of runs is asymptotically normally distributed.

Svensk version:

1 ~~6~~. (25p) Man önskar studera den effektiva livslängden Y (i minuter) för ett skärverktyg på en svarv med hastighet X (varv per minut), där vi har tillgång till två typer av verktyg (A och B).

Vi antar följande modell:

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 D_i + u_i.$$

där

$$D_i = \begin{cases} 1 & \text{om typ B} \\ 0 & \text{om typ A} \end{cases}$$

För data där vi använt 10 verktyg av typ A och 10 verktyg av typ B ($n = 20$), erhölls följande resultat:

ANOVA

Source	Sum of Squares	df
Regression (SSE)	1418.034	2
Residuals (SSR)	157.055	17
Total (SST)	1575.089	19

Coefficients Table

Variable	Coefficient	$SE(\beta_i)$
Constant	36.986	
X	-0.027	0.005
D	15.004	1.36

(a) Beräkna både den ojusterade och justerade (multipla) förklaringsgraden.

(b) Undersök med ett test om minst en av förklaringsvariablerna ska ingå i modellen. Använd lämplig notation och klargör tydligt noll- och mothypotes. Använd signifikansnivå 1%.

(c) Hur tolkar du parametern β_3 i i väntevärdesmening?

(d) Beräkna ett 95% konfidensintervall för β_3.

(e) En alternativ modell är

$$Y_i = \beta_1' + \beta_2' X_i + \beta_3' D_i + \beta_4' X_i D_i + u_i'.$$

Beskriv skillnaden mellan de två modellerna, dvs, i vilken specifik mening är den andra modellen mer flexibel ("rikare")?

(f) Också den alternativa modellen användes för skattning med samma data. Resultat:

ANOVA

Source	Sum of Squares	df
Regression (SSE)	1434.112	3
Residuals (SSR)	140.977	16
Total (SST)	1575.089	19

Coefficients Table

Variable	Coefficient	SE($\hat{\beta}_i$)
Constant	32.775	
X	-0.021	0.0061
D	23.971	6.769
XD	-0.012	0.088

Är den andra modellen signifikant bättre än den första modellen? Använd signifikansnivå 5% med lämpligt test.

2 [redacted]. (25p) Cobb-Douglas produktionsfunktion är

$$Q = \beta_1 L^{\beta_2} K^{\beta_3}, \tag{2}$$

där $L =$" labour input" and $K =$ "capital stock".

Vi vill nu använda $Y = ln(Q)$ som beroende variabel i en linjär regressionsmodell.

(a) Skriv ner en linjär regressionsmodell baserad på (2) och Y definierad enligt ovanstående.

(b) Låt nu $\beta_3 = 1 - \beta_2$. Skriv om den linjära modellen enligt denna restriktion.

(c) Utan att utföra någon beräkning (eftersom vi inte har några numeriska resultat, förutom att vi antar att $n = 33$), beskriv vilket test du skulle använda vid jämförelse av modellen i (a) med modellen i (b). Beskriv i samband med detta nollhypotesen, teststatistikan med dess parametrar och för vilka värden på teststatistikan som nollhypotesen förkastas.

(d) Nämn en potentiell fördel, från ett regressionsmodellperspektiv, med att använda modellen i (b) i stället för modellen i (a).

(e) Genom att använda årsdata för åren 1899 till 1922, fick vi genom att använda modellen i (a) att den residual-baserade skattade autokorrelationskoefficienten $\hat{\rho} = 0.6$. Kan vi från detta dra slutsatsen att vi kan påvisa positiv autokorrelation på (approximativ) signifikansnivå 1%?

38. (20p) För att undersöka den genomsnittliga andelen av inkomsten som en person spenderar på hyran för en lä genhet i New York, samlades data in från 108 singelhushåll. En enkel linjär regressionsmodell skattades med hyran som beroende variabel (Y) och inkomst som oberoende variabel (X). Man misstänker problem med heteroskedasticitet. Whites test användes med följande resultat:

White's Auxiliary Regression

```
Included Observations: 108
 Variable              Coefficient       Std. Error       t-Statistic        Prob.
 C                  -14657900          9288994           -1.577986          0.1176
 INCOME                1200.579          495.1663           2.424598          0.0170
 INCOME²                 -0.010007         0.005355         -1.868714          0.0644
 R-squared                0.082134      Mean dependent var          10515952
 Adjusted R-squared       0.064651      S.D. dependent var          29847739
 S.E. of regression   28866783          Akaike info criterion          37.22167
 Sum squared resid        8.75E + 16    Schwarz criterion              37.29617
 Log likelihood       -2006.970         F-statistic                     4.697874
 Durbin-Watson stat       1.864571      Prob(F-statistic)               0.011115
```

(a) Skriv ner den auxiliära regressionsmodellen som använts i den här situationen för Whites test med lämplig notation.

(b) Visa att vi kan förkasta nollhypotesen om homoskedasticitet på signifikansnivå 5%.

(c) Det visar sig att vi kan modellera variansen för störningsvariabeln som $V(u_i) = \sigma^2 X_i$. Hur kan vi utnyttja detta för att åstadkomma en homoskedastisk modell?

4. (12p) Den "sanna" modellen givet specifik data antas vara

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Av någon anledning används dock följande modell för skattning av β-parametrarna.

$$(Y_i - c_1) = \beta_1 + \beta_2(X_i - c_2) + u_i.$$

där c_1 och c_2 är givna (kända) konstanter.

(a) Härled uttryck för OLS-skattningarna av β_1 och β_2 när "fel" modell används.

(b) Är bägge skattningarna i (a) väntevärdesriktiga?

5. (20p) Sanna eller falska påståenden? Korta motiveringar/kommentarer behövs.

(a) Om X_i i modellen $Y_i = \beta_1 + \beta_2 X_i + u_i$ divideras med 2, ändras både $\hat{\beta}_1$ and $\hat{\beta}_2$.

(b) Om vi i modellen $Y_i = \beta_1 + \beta_2 X_i + u_i$ har mätfel i Y_i, så är $Cov(X_i, u_i) \neq 0$.

(c) Durbin-Watsontestet kan inte användas om vi har autokorrelerade störningsvariabler.

(d) För två stokastiska variabler X_i och Y_i, gäller alltid att $-1 \leq Cov(X_i, Y_i) \leq 1$.

(e) Heteroskedasticitet inträffar när störningsvariabeln i en regressionsmodell är korrellerad med en av de förklarande variablerna.

(f) I Runstestet är antalet "runs" asymptotiskt normalfördelat.

Formula sheet, Econometrics I, Spring 2017

Under the simple linear model $Y_i = \beta_1 + \beta_2 X_i + u_i$, where $u_i \sim N(0, \sigma^2)$ and given independent pairs of observations $(Y_1, X_1), \ldots, (Y_n, X_n)$, the OLS (and ML) estimators are:

$$
\begin{aligned}
\widehat{\beta}_1 &= \overline{Y} - \widehat{\beta}_2 \overline{X} \\
\widehat{\beta}_2 &= \frac{\sum (X_i - \overline{X})(Y_i - \overline{Y})}{\sum (X_i - \overline{X})^2} \\
\widehat{\sigma}^2 &= \frac{RSS}{n-2} = \frac{\sum \left(Y_i - \widehat{Y}_i\right)^2}{n-2}
\end{aligned}
$$

where $\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i$ and where $E(\widehat{\beta_1}) = \beta_1$, $E(\widehat{\beta_2}) = \beta_2$ and $E(\widehat{\sigma}^2) = \sigma^2$ and further

$$
\begin{aligned}
V(\widehat{\beta}_1) &= \frac{\sum X_i^2}{n \sum (X_i - \overline{X})^2} \sigma^2 \\
V(\widehat{\beta}_2) &= \frac{\sigma^2}{\sum (X_i - \overline{X})^2} \\
V(\widehat{Y}_0) &= \sigma^2 \left(\frac{1}{n} + \frac{(X_0 - \overline{X})^2}{\sum (X_i - \overline{X})^2} \right) \\
V(Y_0 - \widehat{Y}_0) &= \sigma^2 \left(1 + \frac{1}{n} + \frac{(X_0 - \overline{X})^2}{\sum (X_i - \overline{X})^2} \right)
\end{aligned}
$$

Distributional results:

$$
\begin{aligned}
\frac{\widehat{\beta}_i - \beta_i}{se(\widehat{\beta}_i)} &\sim t(n-2), \quad i = 1, 2 \\
\frac{\widehat{\sigma}^2 (n-2)}{\sigma^2} &\sim \chi^2(n-2)
\end{aligned}
$$

Coefficient of determination:

$$
r^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum \left(Y_i - \widehat{Y}_i\right)^2}{\sum (Y_i - \overline{Y})^2}
$$

Coefficient of correlation:

$$
r = \frac{\sum (X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum (X_i - \overline{X})^2 \sum (Y_i - \overline{Y})^2}}
$$

where $r = \pm\sqrt{r^2}$

If we let $Y_i^* = w_1 Y_i$ and $X_i^* = w_2 X_2$, then

$$\hat{\beta}_1^* = w_1\hat{\beta}_1, \quad \hat{\beta}_2^* = \left(\frac{w_1}{w_2}\right)\hat{\beta}_2, \quad \hat{\sigma}^{*2} = w_1^2\hat{\sigma}^2$$

Under the multiple linear regression model $Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i$, where $u_i \sim N(0, \sigma^2)$ and given independent vectors of observations $(Y_1, X_{21}, \ldots, X_{k1}), \ldots, (Y_n, X_{2n}, \ldots, X_{kn})$, the following holds for the OLS (ML) estimators:

$$\hat{\sigma}^2 = \frac{RSS}{n-k} = \frac{\sum\left(Y_i - \hat{Y}_i\right)^2}{n-k}$$

$$\frac{\hat{\beta}_i - \beta_i}{se(\hat{\beta}_i)} \sim t(n-k), \quad i = 1, \ldots, k$$

$$\frac{\hat{\sigma}^2\,(n-k)}{\sigma^2} \sim \chi^2(n-k)$$

The multiple coefficient of determination:

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum\left(Y_i - \hat{Y}_i\right)^2}{\sum\left(Y_i - \overline{Y}\right)^2}$$

Adjusted:

$$\overline{R}^2 = 1 - \frac{RSS/(n-k)}{TSS/(n-1)}$$

Testing $H_0 : \beta_2 = \ldots \beta_k = 0$:

$$F = \frac{ESS/(k-1)}{RSS/(n-k)} = \frac{\sum\left(\hat{Y}_i - \overline{Y}\right)^2/(k-1)}{\sum\left(Y_i - \hat{Y}_i\right)^2/(n-k)}$$

Comparing an "old" model with a "new" (larger):

$$\begin{aligned} F &= \frac{(ESS_{new} - ESS_{old})/\text{number of new regressors}}{RSS_{new}/(n - \text{number of parameters in the new model})} \\ &= \frac{(R^2_{new} - R^2_{old})/\text{number of new regressors}}{(1 - R^2_{new})/(n - \text{number of parameters in the new model})} \end{aligned}$$

Comparing an "unrestricted" model with a "restricted":

$$F = \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)} = \frac{(R^2_{UR} - R^2_R)/m}{(1 - R^2_{UR})/(n-k)}$$

where m is the number of linear constraints and k is the number of parameters in the unrestricted model.

Variance inflation factor:

$$VIF_j = \frac{1}{1 - R_j^2}$$

Auxiliary regression:

$$F_j = \frac{R_j^2/(k-2)}{(1 - R_j^2)/(n-k+1)}$$

where $R_j^2 = R^2$ in the regression of the remaining $(k-2)$ regressors.

White's test for heteroscedasticity:

$$n R^2 \overset{\text{appr}}{\sim} \chi^2(df = \text{number of regressors in the auxiliary regression})$$

(Holds under H_0 : no heteroscedasticity.)

For R = number of runs, where $N = N_1 + N_2$ total number of observations:

$$E(R) = \frac{2N_1N_2}{N} + 1$$

$$V(R) = \frac{2N_1N_2(2N_1N_2 - N)}{N^2(N-1)}$$

The Durbin Watson d statistic:

$$d = \frac{\sum_{t=2}^{n} (\widehat{u}_t - \widehat{u}_{t-1})^2}{\sum_{t=1}^{n} \widehat{u}_t^2}$$

Akaike's information criterion:

$$AIC = \frac{e^{2k/n} RSS}{n}$$

Schwartz's information citerion:

$$SIC = \frac{n^{k/n} RSS}{n}$$

Mallow's C_p criterion:

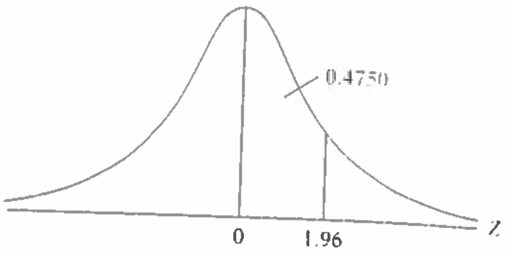
$$C_p = \frac{RSS_p}{\widehat{\sigma}^2} - (n - 2p)$$

TABLE D.1
Areas Under the Standardized Normal Distribution

Example

$\Pr(0 \leq Z \leq 1.96) = 0.4750$

$\Pr(Z \geq 1.96) = 0.5 - 0.4750 = 0.025$



Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4454	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Note. This table gives the area in the right-hand tail of the distribution (i.e., $Z \geq 0$). But since the normal distribution is symmetrical about $Z = 0$, the area in the left-hand tail is the same as the area in the corresponding right-hand tail. For example, $P(-1.96 \leq Z \leq 0) = 0.4750$. Therefore, $P(-1.96 \leq Z \leq 1.96) = 2(0.4750) = 0.95$.

TABLE D.2

Percentage Points of the *t* Distribution

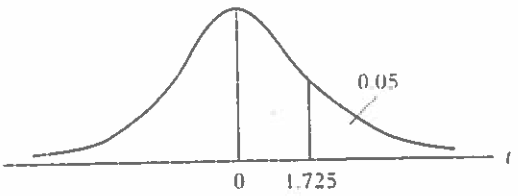
Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



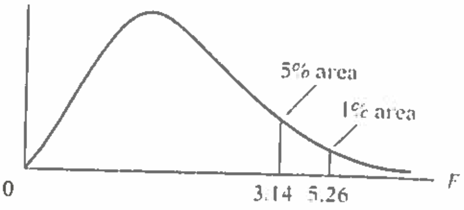
df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

TABLE D.3 Upper Percentage Points of the F Distribution

Example

$Pr(F > 1.59) = 0.25$
$Pr(F > 2.42) = 0.10$ for df $N_1 = 10$
$Pr(F > 3.14) = 0.05$ and $N_2 = 9$
$Pr(F > 5.26) = 0.01$



df for denominator N_2		df for numerator N_1											
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
1	.25	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.36	9.41
	.10	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	60.5	60.7
	.05	161	200	216	225	230	234	237	239	241	242	243	244
2	.25	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.39
	.10	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41
	.05	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
	.01	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
3	.25	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.45
	.10	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22
	.05	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
	.01	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1
4	.25	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08
	.10	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.91	3.90
	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
	.01	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.4
5	.25	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89
	.10	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.28	3.27
	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.71	4.68
	.01	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89
6	.25	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.77
	.10	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.92	2.90
	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
	.01	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72
7	.25	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.69	1.68
	.10	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.68	2.67
	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
	.01	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47
8	.25	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.63	1.63	1.63	1.62
	.10	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50
	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
	.01	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67
9	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
	.01	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 18, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

F-table continued

df for numerator N_1													df for denominator N_2
15	20	24	30	40	50	60	100	120	200	500	∞	Pr	
9.49	9.58	9.63	9.67	9.71	9.74	9.76	9.78	9.80	9.82	9.84	9.85	.25	1
61.2	61.7	62.0	62.3	62.5	62.7	62.8	63.0	63.1	63.2	63.3	63.3	.10	
246	248	249	250	251	252	252	253	253	254	254	254	.05	
3.41	3.43	3.43	3.44	3.45	3.45	3.46	3.47	3.47	3.48	3.48	3.48	.25	2
9.42	9.44	9.45	9.46	9.47	9.47	9.47	9.48	9.48	9.49	9.49	9.49	.10	
19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	.05	
99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	.01	
2.46	2.46	2.46	2.47	2.47	2.47	2.47	2.47	2.47	2.47	2.47	2.47	.25	3
5.20	5.18	5.18	5.17	5.16	5.15	5.15	5.14	5.14	5.14	5.14	5.13	.10	
8.70	8.66	8.64	8.62	8.59	8.58	8.57	8.55	8.55	8.54	8.53	8.53	.05	
26.9	26.7	26.6	26.5	26.4	26.4	26.3	26.2	26.2	26.2	26.1	26.1	.01	
2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	.25	4
3.87	3.84	3.83	3.82	3.80	3.80	3.79	3.78	3.78	3.77	3.76	3.76	.10	
5.86	5.80	5.77	5.75	5.72	5.70	5.69	5.66	5.66	5.65	5.64	5.63	.05	
14.2	14.0	13.9	13.8	13.7	13.7	13.7	13.6	13.6	13.5	13.5	13.5	.01	
1.89	1.88	1.88	1.88	1.88	1.88	1.87	1.87	1.87	1.87	1.87	1.87	.25	5
3.24	3.21	3.19	3.17	3.16	3.15	3.14	3.13	3.12	3.12	3.11	3.10	.10	
4.62	4.56	4.53	4.50	4.46	4.44	4.43	4.41	4.40	4.39	4.37	4.36	.05	
9.72	9.55	9.47	9.38	9.29	9.24	9.20	9.13	9.11	9.08	9.04	9.02	.01	
1.76	1.76	1.75	1.75	1.75	1.75	1.74	1.74	1.74	1.74	1.74	1.74	.25	6
2.87	2.84	2.82	2.80	2.78	2.77	2.76	2.75	2.74	2.73	2.73	2.72	.10	
3.94	3.87	3.84	3.81	3.77	3.75	3.74	3.71	3.70	3.69	3.68	3.67	.05	
7.56	7.40	7.31	7.23	7.14	7.09	7.06	6.99	6.97	6.93	6.90	6.88	.01	
1.68	1.67	1.67	1.66	1.66	1.66	1.65	1.65	1.65	1.65	1.65	1.65	.25	7
2.63	2.59	2.58	2.56	2.54	2.52	2.51	2.50	2.49	2.48	2.48	2.47	.10	
3.51	3.44	3.41	3.38	3.34	3.32	3.30	3.27	3.27	3.25	3.24	3.23	.05	
6.31	6.16	6.07	5.99	5.91	5.86	5.82	5.75	5.74	5.70	5.67	5.65	.01	
1.62	1.61	1.60	1.60	1.59	1.59	1.59	1.58	1.58	1.58	1.58	1.58	.25	8
2.46	2.42	2.40	2.38	2.36	2.35	2.34	2.32	2.32	2.31	2.30	2.29	.10	
3.22	3.15	3.12	3.08	3.04	2.02	3.01	2.97	2.97	2.95	2.94	2.93	.05	
5.52	5.36	5.28	5.20	5.12	5.07	5.03	4.96	4.95	4.91	4.88	4.86	.01	
1.57	1.56	1.56	1.55	1.55	1.54	1.54	1.53	1.53	1.53	1.53	1.53	.25	9
2.34	2.30	2.28	2.25	2.23	2.22	2.21	2.19	2.18	2.17	2.17	2.16	.10	
3.01	2.94	2.90	2.86	2.83	2.80	2.79	2.76	2.75	2.73	2.72	2.71	.05	
4.96	4.81	4.73	4.65	4.57	4.52	4.48	4.42	4.40	4.36	4.33	4.31	.01	

(Continued)

TABLE D.3 Upper Percentage Points of the F Distribution *(Continued)*

df for denominator N_2		df for numerator N_1											
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
10	.25	1.49	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.55	1.54
	.10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.30	2.28
	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
	.01	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71
11	.25	1.47	1.58	1.58	1.57	1.56	1.55	1.54	1.53	1.53	1.52	1.52	1.51
	.10	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.23	2.21
	.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
	.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40
12	.25	1.46	1.56	1.56	1.55	1.54	1.53	1.52	1.51	1.51	1.50	1.50	1.49
	.10	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.17	2.15
	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16
13	.25	1.45	1.55	1.55	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.47
	.10	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.12	2.10
	.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
	.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96
14	.25	1.44	1.53	1.53	1.52	1.51	1.50	1.49	1.48	1.47	1.46	1.46	1.45
	.10	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.08	2.05
	.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
	.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.86	3.80
15	.25	1.43	1.52	1.52	1.51	1.49	1.48	1.47	1.46	1.46	1.45	1.44	1.44
	.10	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.04	2.02
	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67
16	.25	1.42	1.51	1.51	1.50	1.48	1.47	1.46	1.45	1.44	1.44	1.44	1.43
	.10	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	2.01	1.99
	.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42
	.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55
17	.25	1.42	1.51	1.50	1.49	1.47	1.46	1.45	1.44	1.43	1.43	1.42	1.41
	.10	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.98	1.96
	.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38
	.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46
18	.25	1.41	1.50	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.42	1.41	1.40
	.10	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.96	1.93
	.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34
	.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37
19	.25	1.41	1.49	1.49	1.47	1.46	1.44	1.43	1.42	1.41	1.41	1.40	1.40
	.10	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.94	1.91
	.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31
	.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30
20	.25	1.40	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.41	1.40	1.39	1.39
	.10	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.92	1.89
	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23

F-table continued

df for numerator N_1													df for denominator N_2
15	20	24	30	40	50	60	100	120	200	500	∞	Pr	
1.53	1.52	1.52	1.51	1.51	1.50	1.50	1.49	1.49	1.49	1.48	1.48	.25	10
2.24	2.20	2.18	2.16	2.13	2.12	2.11	2.09	2.08	2.07	2.06	2.06	.10	
2.85	2.77	2.74	2.70	2.66	2.64	2.62	2.59	2.58	2.56	2.55	2.54	.05	
4.56	4.41	4.33	4.25	4.17	4.12	4.08	4.01	4.00	3.96	3.93	3.91	.01	
1.50	1.49	1.49	1.48	1.47	1.47	1.47	1.46	1.46	1.46	1.45	1.45	.25	11
2.17	2.12	2.10	2.08	2.05	2.04	2.03	2.00	2.00	1.99	1.98	1.97	.10	
2.72	2.65	2.61	2.57	2.53	2.51	2.49	2.46	2.45	2.43	2.42	2.40	.05	
4.25	4.10	4.02	3.94	3.86	3.81	3.78	3.71	3.69	3.66	3.62	3.60	.01	
1.48	1.47	1.46	1.45	1.45	1.44	1.44	1.43	1.43	1.43	1.42	1.42	.25	12
2.10	2.06	2.04	2.01	1.99	1.97	1.96	1.94	1.93	1.92	1.91	1.90	.10	
2.62	2.54	2.51	2.47	2.43	2.40	2.38	2.35	2.34	2.32	2.31	2.30	.05	
4.01	3.86	3.78	3.70	3.62	3.57	3.54	3.47	3.45	3.41	3.38	3.36	.01	
1.46	1.45	1.44	1.43	1.42	1.42	1.42	1.41	1.41	1.40	1.40	1.40	.25	13
2.05	2.01	1.98	1.96	1.93	1.92	1.90	1.88	1.88	1.86	1.85	1.85	.10	
2.53	2.46	2.42	2.38	2.34	2.31	2.30	2.26	2.25	2.23	2.22	2.21	.05	
3.82	3.66	3.59	3.51	3.43	3.38	3.34	3.27	3.25	3.22	3.19	3.17	.01	
1.44	1.43	1.42	1.41	1.41	1.40	1.40	1.39	1.39	1.39	1.38	1.38	.25	14
2.01	1.96	1.94	1.91	1.89	1.87	1.86	1.83	1.83	1.82	1.80	1.80	.10	
2.46	2.39	2.35	2.31	2.27	2.24	2.22	2.19	2.18	2.16	2.14	2.13	.05	
3.66	3.51	3.43	3.35	3.27	3.22	3.18	3.11	3.09	3.06	3.03	3.00	.01	
1.43	1.41	1.41	1.40	1.39	1.39	1.38	1.38	1.37	1.37	1.36	1.36	.25	15
1.97	1.92	1.90	1.87	1.85	1.83	1.82	1.79	1.79	1.77	1.76	1.76	.10	
2.40	2.33	2.29	2.25	2.20	2.18	2.16	2.12	2.11	2.10	2.08	2.07	.05	
3.52	3.37	3.29	3.21	3.13	3.08	3.05	2.98	2.96	2.92	2.89	2.87	.01	
1.41	1.40	1.39	1.38	1.37	1.37	1.36	1.36	1.35	1.35	1.34	1.34	.25	16
1.94	1.89	1.87	1.84	1.81	1.79	1.78	1.76	1.75	1.74	1.73	1.72	.10	
2.35	2.28	2.24	2.19	2.15	2.12	2.11	2.07	2.06	2.04	2.02	2.01	.05	
3.41	3.26	3.18	3.10	3.02	2.97	2.93	2.86	2.84	2.81	2.78	2.75	.01	
1.40	1.39	1.38	1.37	1.36	1.35	1.35	1.34	1.34	1.34	1.33	1.33	.25	17
1.91	1.86	1.84	1.81	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.69	.10	
2.31	2.23	2.19	2.15	2.10	2.08	2.06	2.02	2.01	1.99	1.97	1.96	.05	
3.31	3.16	3.08	3.00	2.92	2.87	2.83	2.76	2.75	2.71	2.68	2.65	.01	
1.39	1.38	1.37	1.36	1.35	1.34	1.34	1.33	1.33	1.32	1.32	1.32	.25	18
1.89	1.84	1.81	1.78	1.75	1.74	1.72	1.70	1.69	1.68	1.67	1.66	.10	
2.27	2.19	2.15	2.11	2.06	2.04	2.02	1.98	1.97	1.95	1.93	1.92	.05	
3.23	3.08	3.00	2.92	2.84	2.78	2.75	2.68	2.66	2.62	2.59	2.57	.01	
1.38	1.37	1.36	1.35	1.34	1.33	1.33	1.32	1.32	1.31	1.31	1.30	.25	19
1.86	1.81	1.79	1.76	1.73	1.71	1.70	1.67	1.67	1.65	1.64	1.63	.10	
2.23	2.16	2.11	2.07	2.03	2.00	1.98	1.94	1.93	1.91	1.89	1.88	.05	
3.15	3.00	2.92	2.84	2.76	2.71	2.67	2.60	2.58	2.55	2.51	2.49	.01	
1.37	1.36	1.35	1.34	1.33	1.33	1.32	1.31	1.31	1.30	1.30	1.29	.25	20
1.84	1.79	1.77	1.74	1.71	1.69	1.68	1.65	1.64	1.63	1.62	1.61	.10	
2.20	2.12	2.08	2.04	1.99	1.97	1.95	1.91	1.90	1.88	1.86	1.84	.05	
3.09	2.94	2.86	2.78	2.69	2.64	2.61	2.54	2.52	2.48	2.44	2.42	.01	

(Continued

TABLE D.3 **Upper Percentage Points of the *F* Distribution** (*Continued*)

df for denominator N_2		df for numerator N_1											
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
22	.25	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.38	1.37
	.10	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86
	.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12
24	.25	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36
	.10	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83
	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03
26	.25	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.36	1.35
	.10	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.84	1.81
	.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96
28	.25	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	.10	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79
	.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90
30	.25	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.35	1.34
	.10	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77
	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84
40	.25	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31
	.10	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.73	1.71
	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66
60	.25	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.29
	.10	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66
	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50
120	.25	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.27	1.26
	.10	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.62	1.60
	.05	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.87	1.83
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34
200	.25	1.33	1.39	1.38	1.36	1.34	1.32	1.31	1.29	1.28	1.27	1.26	1.25
	.10	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.60	1.57
	.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27
∞	.25	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.24
	.10	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55
	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18

F-table continued

df for numerator N_1													df for denominator N_2
15	20	24	30	40	50	60	100	120	200	500	∞	Pr	
1.36	1.34	1.33	1.32	1.31	1.31	1.30	1.30	1.30	1.29	1.29	1.28	.25	22
1.81	1.76	1.73	1.70	1.67	1.65	1.64	1.61	1.60	1.59	1.58	1.57	.10	
2.15	2.07	2.03	1.98	1.94	1.91	1.89	1.85	1.84	1.82	1.80	1.78	.05	
2.98	2.83	2.75	2.67	2.58	2.53	2.50	2.42	2.40	2.36	2.33	2.31	.01	
1.35	1.33	1.32	1.31	1.30	1.29	1.29	1.28	1.28	1.27	1.27	1.26	.25	24
1.78	1.73	1.70	1.67	1.64	1.62	1.61	1.58	1.57	1.56	1.54	1.53	.10	
2.11	2.03	1.98	1.94	1.89	1.86	1.84	1.80	1.79	1.77	1.75	1.73	.05	
2.89	2.74	2.66	2.58	2.49	2.44	2.40	2.33	2.31	2.27	2.24	2.21	.01	
1.34	1.32	1.31	1.30	1.29	1.28	1.28	1.26	1.26	1.26	1.25	1.25	.25	26
1.76	1.71	1.68	1.65	1.61	1.59	1.58	1.55	1.54	1.53	1.51	1.50	.10	
2.07	1.99	1.95	1.90	1.85	1.82	1.80	1.76	1.75	1.73	1.71	1.69	.05	
2.81	2.66	2.58	2.50	2.42	2.36	2.33	2.25	2.23	2.19	2.16	2.13	.01	
1.33	1.31	1.30	1.29	1.28	1.27	1.27	1.26	1.25	1.25	1.24	1.24	.25	28
1.74	1.69	1.66	1.63	1.59	1.57	1.56	1.53	1.52	1.50	1.49	1.48	.10	
2.04	1.96	1.91	1.87	1.82	1.79	1.77	1.73	1.71	1.69	1.67	1.65	.05	
2.75	2.60	2.52	2.44	2.35	2.30	2.26	2.19	2.17	2.13	2.09	2.06	.01	
1.32	1.30	1.29	1.28	1.27	1.26	1.26	1.25	1.24	1.24	1.23	1.23	.25	30
1.72	1.67	1.64	1.61	1.57	1.55	1.54	1.51	1.50	1.48	1.47	1.46	.10	
2.01	1.93	1.89	1.84	1.79	1.76	1.74	1.70	1.68	1.66	1.64	1.62	.05	
2.70	2.55	2.47	2.39	2.30	2.25	2.21	2.13	2.11	2.07	2.03	2.01	.01	
1.30	1.28	1.26	1.25	1.24	1.23	1.22	1.21	1.21	1.20	1.19	1.19	.25	40
1.66	1.61	1.57	1.54	1.51	1.48	1.47	1.43	1.42	1.41	1.39	1.38	.10	
1.92	1.84	1.79	1.74	1.69	1.66	1.64	1.59	1.58	1.55	1.53	1.51	.05	
2.52	2.37	2.29	2.20	2.11	2.06	2.02	1.94	1.92	1.87	1.83	1.80	.01	
1.27	1.25	1.24	1.22	1.21	1.20	1.19	1.17	1.17	1.16	1.15	1.15	.25	60
1.60	1.54	1.51	1.48	1.44	1.41	1.40	1.36	1.35	1.33	1.31	1.29	.10	
1.84	1.75	1.70	1.65	1.59	1.56	1.53	1.48	1.47	1.44	1.41	1.39	.05	
2.35	2.20	2.12	2.03	1.94	1.88	1.84	1.75	1.73	1.68	1.63	1.60	.01	
1.24	1.22	1.21	1.19	1.18	1.17	1.16	1.14	1.13	1.12	1.11	1.10	.25	120
1.55	1.48	1.45	1.41	1.37	1.34	1.32	1.27	1.26	1.24	1.21	1.19	.10	
1.75	1.66	1.61	1.55	1.50	1.46	1.43	1.37	1.35	1.32	1.28	1.25	.05	
2.19	2.03	1.95	1.86	1.76	1.70	1.66	1.56	1.53	1.48	1.42	1.38	.01	
1.23	1.21	1.20	1.18	1.16	1.14	1.12	1.11	1.10	1.09	1.08	1.06	.25	200
1.52	1.46	1.42	1.38	1.34	1.31	1.28	1.24	1.22	1.20	1.17	1.14	.10	
1.72	1.62	1.57	1.52	1.46	1.41	1.39	1.32	1.29	1.26	1.22	1.19	.05	
2.13	1.97	1.89	1.79	1.69	1.63	1.58	1.48	1.44	1.39	1.33	1.28	.01	
1.22	1.19	1.18	1.16	1.14	1.13	1.12	1.09	1.08	1.07	1.04	1.00	.25	∞
1.49	1.42	1.38	1.34	1.30	1.26	1.24	1.18	1.17	1.13	1.08	1.00	.10	
1.67	1.57	1.52	1.46	1.39	1.35	1.32	1.24	1.22	1.17	1.11	1.00	.05	
2.04	1.88	1.79	1.70	1.59	1.52	1.47	1.36	1.32	1.25	1.15	1.00	.01	

TABLE D.4

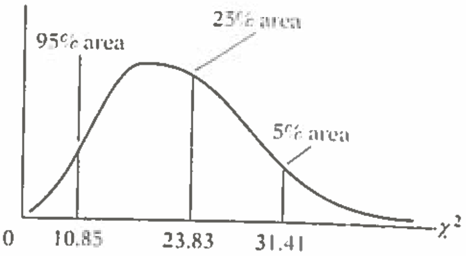
Upper Percentage Points of the χ^2 Distribution

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



Degrees of freedom \ Pr	.995	.990	.975	.950	.900
1	392704×10^{-10}	157088×10^{-9}	982069×10^{-9}	393214×10^{-8}	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100*	67.3276	70.0648	74.2219	77.9295	82.3581

*For df greater than 100 the expression $\sqrt{2\chi^2} - \sqrt{(2k - 1)} = Z$ follows the standardized normal distribution, where k represents the degrees of freedom

χ^2-table continued

.750	.500	.250	.100	.050	.025	.010	.005
.1015308	.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
.575364	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
1.212534	2.36597	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381
1.92255	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
2.67460	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
3.45460	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
4.25485	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
5.07064	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
5.89883	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
6.73720	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
7.58412	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
8.43842	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
9.29906	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
10.1653	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193
11.0365	14.3389	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013
11.9122	15.3385	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672
12.7919	16.3381	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185
13.6753	17.3379	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564
14.5620	18.3376	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822
15.4518	19.3374	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968
16.3444	20.3372	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010
17.2396	21.3370	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956
18.1373	22.3369	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813
19.0372	23.3367	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585
19.9393	24.3366	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278
20.8434	25.3364	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899
21.7494	26.3363	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449
22.6572	27.3363	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933
23.5666	28.3362	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356
4.4776	29.3360	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720
33.6603	39.3354	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659
42.9421	49.3349	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900
52.2938	59.3347	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517
61.6983	69.3344	77.5766	85.5271	90.5312	95.0231	100.425	104.215
71.1445	79.3343	88.1303	96.5782	101.879	106.629	112.329	116.321
80.6247	89.3342	98.6499	107.565	113.145	118.136	124.116	128.299
90.1332	99.3341	109.141	118.498	124.342	129.561	135.807	140.169

Source: Abridged from E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 8, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

TABLE D.5A **Durbin–Watson *d* Statistic: Significance Points of d_L and d_U at 0.05 Level of Significance**

	$k' = 1$		$k' = 2$		$k' = 3$		$k' = 4$		$k' = 5$		$k' = 6$		$k' = 7$		$k' = 8$		$k' = 9$		$k' = 10$	
n	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
6	0.610	1.400	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	0.700	1.356	0.467	1.896	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8	0.763	1.332	0.559	1.777	0.368	2.287	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9	0.824	1.320	0.629	1.699	0.455	2.128	0.296	2.588	—	—	—	—	—	—	—	—	—	—	—	—
10	0.879	1.320	0.697	1.641	0.525	2.016	0.376	2.414	0.243	2.822	—	—	—	—	—	—	—	—	—	—
11	0.927	1.324	0.658	1.604	0.595	1.928	0.444	2.283	0.316	2.645	0.203	3.005	—	—	—	—	—	—	—	—
12	0.971	1.331	0.812	1.579	0.658	1.864	0.512	2.177	0.379	2.506	0.268	2.832	0.171	3.149	—	—	—	—	—	—
13	1.010	1.340	0.861	1.562	0.715	1.816	0.574	2.094	0.445	2.390	0.328	2.692	0.230	2.985	0.147	3.266	—	—	—	—
14	1.045	1.350	0.905	1.551	0.767	1.779	0.632	2.030	0.505	2.296	0.389	2.572	0.286	2.848	0.200	3.111	0.127	3.360	—	—
15	1.077	1.361	0.946	1.543	0.814	1.750	0.685	1.977	0.562	2.220	0.447	2.472	0.343	2.727	0.251	2.979	0.175	3.216	0.111	3.438
16	1.106	1.371	0.982	1.539	0.857	1.728	0.734	1.935	0.615	2.157	0.502	2.388	0.398	2.624	0.304	2.860	0.222	3.090	0.155	3.304
17	1.133	1.381	1.015	1.536	0.897	1.710	0.779	1.900	0.664	2.104	0.554	2.318	0.451	2.537	0.356	2.757	0.272	2.975	0.198	3.184
18	1.158	1.391	1.046	1.535	0.933	1.696	0.820	1.872	0.710	2.060	0.603	2.257	0.502	2.461	0.407	2.667	0.321	2.873	0.244	3.073
19	1.180	1.401	1.074	1.536	0.967	1.685	0.859	1.848	0.752	2.023	0.649	2.206	0.549	2.396	0.456	2.589	0.369	2.783	0.290	2.974
20	1.201	1.411	1.100	1.537	0.998	1.676	0.894	1.828	0.792	1.991	0.692	2.162	0.595	2.339	0.502	2.521	0.416	2.704	0.336	2.885
21	1.221	1.420	1.125	1.538	1.026	1.669	0.927	1.812	0.829	1.964	0.732	2.124	0.637	2.290	0.547	2.460	0.461	2.633	0.380	2.806
22	1.239	1.429	1.147	1.541	1.053	1.664	0.958	1.797	0.863	1.940	0.769	2.090	0.677	2.246	0.588	2.407	0.504	2.571	0.424	2.734
23	1.257	1.437	1.168	1.543	1.078	1.660	0.986	1.785	0.895	1.920	0.804	2.061	0.715	2.208	0.628	2.360	0.545	2.514	0.465	2.670
24	1.273	1.446	1.188	1.546	1.101	1.656	1.013	1.775	0.925	1.902	0.837	2.035	0.751	2.174	0.666	2.318	0.584	2.464	0.506	2.613
25	1.288	1.454	1.206	1.550	1.123	1.654	1.038	1.767	0.953	1.886	0.868	2.012	0.784	2.144	0.702	2.280	0.621	2.419	0.544	2.560
26	1.302	1.461	1.224	1.553	1.143	1.652	1.062	1.759	0.979	1.873	0.897	1.992	0.816	2.117	0.735	2.246	0.657	2.379	0.581	2.513
27	1.316	1.469	1.240	1.556	1.162	1.651	1.084	1.753	1.004	1.861	0.925	1.974	0.845	2.093	0.767	2.216	0.691	2.342	0.616	2.470
28	1.328	1.476	1.255	1.560	1.181	1.650	1.104	1.747	1.028	1.850	0.951	1.958	0.874	2.071	0.798	2.188	0.723	2.309	0.650	2.431
29	1.341	1.483	1.270	1.563	1.198	1.650	1.124	1.743	1.050	1.841	0.975	1.944	0.900	2.052	0.826	2.164	0.753	2.278	0.682	2.396
30	1.352	1.489	1.284	1.567	1.214	1.650	1.143	1.739	1.071	1.833	0.998	1.931	0.926	2.034	0.854	2.141	0.782	2.251	0.712	2.363
31	1.363	1.496	1.297	1.570	1.229	1.650	1.160	1.735	1.090	1.825	1.020	1.920	0.950	2.018	0.879	2.120	0.810	2.226	0.741	2.333
32	1.373	1.502	1.309	1.574	1.244	1.650	1.177	1.732	1.109	1.819	1.041	1.909	0.972	2.004	0.904	2.102	0.836	2.203	0.769	2.306
33	1.383	1.508	1.321	1.577	1.258	1.651	1.193	1.730	1.127	1.813	1.061	1.900	0.994	1.991	0.927	2.085	0.861	2.181	0.795	2.281
34	1.393	1.514	1.333	1.580	1.271	1.652	1.208	1.728	1.144	1.808	1.080	1.891	1.015	1.979	0.950	2.069	0.885	2.162	0.821	2.257
35	1.402	1.519	1.343	1.584	1.283	1.653	1.222	1.726	1.160	1.803	1.097	1.884	1.034	1.967	0.971	2.054	0.908	2.144	0.845	2.236
36	1.411	1.525	1.354	1.587	1.295	1.654	1.236	1.724	1.175	1.799	1.114	1.877	1.053	1.957	0.991	2.041	0.930	2.127	0.868	2.216
37	1.419	1.530	1.364	1.590	1.307	1.655	1.249	1.723	1.190	1.795	1.131	1.870	1.071	1.948	1.011	2.029	0.951	2.112	0.891	2.198
38	1.427	1.535	1.373	1.594	1.318	1.656	1.261	1.722	1.204	1.792	1.146	1.864	1.088	1.939	1.029	2.017	0.970	2.098	0.912	2.180
39	1.435	1.540	1.382	1.597	1.328	1.658	1.273	1.722	1.218	1.789	1.161	1.859	1.104	1.932	1.047	2.007	0.990	2.085	0.932	2.164
40	1.442	1.544	1.391	1.600	1.338	1.659	1.285	1.721	1.230	1.786	1.175	1.854	1.120	1.924	1.064	1.997	1.008	2.072	0.952	2.149
45	1.475	1.566	1.430	1.615	1.383	1.666	1.336	1.720	1.287	1.776	1.238	1.835	1.189	1.895	1.139	1.958	1.089	2.022	1.038	2.088
50	1.503	1.585	1.462	1.628	1.421	1.674	1.378	1.721	1.335	1.771	1.291	1.822	1.246	1.875	1.201	1.930	1.156	1.986	1.110	2.044
55	1.528	1.601	1.490	1.641	1.452	1.681	1.414	1.724	1.374	1.768	1.334	1.814	1.294	1.861	1.253	1.909	1.212	1.959	1.170	2.010
60	1.549	1.616	1.514	1.652	1.480	1.689	1.444	1.727	1.408	1.767	1.372	1.808	1.335	1.850	1.298	1.894	1.260	1.939	1.222	1.984
65	1.567	1.629	1.536	1.662	1.503	1.696	1.471	1.731	1.438	1.767	1.404	1.805	1.370	1.843	1.336	1.882	1.301	1.923	1.266	1.964
70	1.583	1.641	1.554	1.672	1.525	1.703	1.494	1.735	1.464	1.768	1.433	1.802	1.401	1.837	1.369	1.873	1.337	1.910	1.305	1.948
75	1.598	1.652	1.571	1.680	1.543	1.709	1.515	1.739	1.487	1.770	1.458	1.801	1.428	1.834	1.399	1.867	1.369	1.901	1.339	1.935
80	1.611	1.662	1.586	1.688	1.560	1.715	1.534	1.743	1.507	1.772	1.480	1.801	1.453	1.831	1.425	1.861	1.397	1.893	1.369	1.925
85	1.624	1.671	1.600	1.696	1.575	1.721	1.550	1.747	1.525	1.774	1.500	1.801	1.474	1.829	1.448	1.857	1.422	1.886	1.396	1.916
90	1.635	1.679	1.612	1.703	1.589	1.726	1.566	1.751	1.542	1.776	1.518	1.801	1.494	1.827	1.469	1.854	1.445	1.881	1.420	1.909
95	1.645	1.687	1.623	1.709	1.602	1.732	1.579	1.755	1.557	1.778	1.535	1.802	1.512	1.827	1.489	1.852	1.465	1.877	1.442	1.903
100	1.654	1.694	1.634	1.715	1.613	1.736	1.592	1.758	1.571	1.780	1.550	1.803	1.528	1.826	1.506	1.850	1.484	1.874	1.462	1.898
150	1.720	1.746	1.706	1.760	1.693	1.774	1.679	1.788	1.665	1.802	1.651	1.817	1.637	1.832	1.622	1.847	1.608	1.862	1.594	1.877
200	1.758	1.778	1.748	1.789	1.738	1.799	1.728	1.810	1.718	1.820	1.707	1.831	1.697	1.841	1.686	1.852	1.675	1.863	1.665	1.874

	$k' = 11$		$k' = 12$		$k' = 13$		$k' = 14$		$k' = 15$		$k' = 16$		$k' = 17$		$k' = 18$		$k' = 19$		$k' = 20$	
n	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
16	0.098	3.503	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	0.138	3.378	0.087	3.557	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	0.177	3.265	0.123	3.441	0.078	3.603	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	0.220	3.159	0.160	3.335	0.111	3.496	0.070	3.642	—	—	—	—	—	—	—	—	—	—	—	—
20	0.263	3.063	0.200	3.234	0.145	3.395	0.100	3.542	0.063	3.676	—	—	—	—	—	—	—	—	—	—
21	0.307	2.976	0.240	3.141	0.182	3.300	0.132	3.448	0.091	3.583	0.058	3.705	—	—	—	—	—	—	—	—
22	0.349	2.897	0.281	3.057	0.220	3.211	0.166	3.358	0.120	3.495	0.083	3.619	0.052	3.731	—	—	—	—	—	—
23	0.391	2.826	0.322	2.979	0.259	3.128	0.202	3.272	0.153	3.409	0.110	3.535	0.076	3.650	0.048	3.753	—	—	—	—
24	0.431	2.761	0.362	2.908	0.297	3.053	0.239	3.193	0.186	3.327	0.141	3.454	0.101	3.572	0.070	3.678	0.044	3.773	—	—
25	0.470	2.702	0.400	2.844	0.335	2.983	0.275	3.119	0.221	3.251	0.172	3.376	0.130	3.494	0.094	3.604	0.065	3.702	0.041	3.790
26	0.508	2.649	0.438	2.784	0.373	2.919	0.312	3.051	0.256	3.179	0.205	3.303	0.160	3.420	0.120	3.531	0.087	3.632	0.060	3.724
27	0.544	2.600	0.475	2.730	0.409	2.859	0.348	2.987	0.291	3.112	0.238	3.233	0.191	3.349	0.149	3.460	0.112	3.563	0.081	3.658
28	0.578	2.555	0.510	2.680	0.445	2.805	0.383	2.928	0.325	3.050	0.271	3.168	0.222	3.283	0.178	3.392	0.138	3.495	0.104	3.592
29	0.612	2.515	0.544	2.634	0.479	2.755	0.418	2.874	0.359	2.992	0.305	3.107	0.254	3.219	0.208	3.327	0.166	3.431	0.129	3.528
30	0.643	2.477	0.577	2.592	0.512	2.708	0.451	2.823	0.392	2.937	0.337	3.050	0.286	3.160	0.238	3.266	0.195	3.368	0.156	3.465
31	0.674	2.443	0.608	2.553	0.545	2.665	0.484	2.776	0.425	2.887	0.370	2.996	0.317	3.103	0.269	3.208	0.224	3.309	0.183	3.406
32	0.703	2.411	0.638	2.517	0.576	2.625	0.515	2.733	0.457	2.840	0.401	2.946	0.349	3.050	0.299	3.153	0.253	3.252	0.211	3.348
33	0.731	2.382	0.668	2.484	0.606	2.588	0.546	2.692	0.488	2.796	0.432	2.899	0.379	3.000	0.329	3.100	0.283	3.198	0.239	3.293
34	0.758	2.355	0.695	2.454	0.634	2.554	0.575	2.654	0.518	2.754	0.462	2.854	0.409	2.954	0.359	3.051	0.312	3.147	0.267	3.240
35	0.783	2.330	0.722	2.425	0.662	2.521	0.604	2.619	0.547	2.716	0.492	2.813	0.439	2.910	0.388	3.005	0.340	3.099	0.295	3.190
36	0.808	2.306	0.748	2.398	0.689	2.492	0.631	2.586	0.575	2.680	0.520	2.774	0.467	2.868	0.417	2.961	0.369	3.053	0.323	3.142
37	0.831	2.285	0.772	2.374	0.714	2.464	0.657	2.555	0.602	2.646	0.548	2.738	0.495	2.829	0.445	2.920	0.397	3.009	0.351	3.097
38	0.854	2.265	0.796	2.351	0.739	2.438	0.683	2.526	0.628	2.614	0.575	2.703	0.522	2.792	0.472	2.880	0.424	2.968	0.378	3.054
39	0.875	2.246	0.819	2.329	0.763	2.413	0.707	2.499	0.653	2.585	0.600	2.671	0.549	2.757	0.499	2.843	0.451	2.929	0.404	3.013
40	0.896	2.228	0.840	2.309	0.785	2.391	0.731	2.473	0.678	2.557	0.626	2.641	0.575	2.724	0.525	2.808	0.477	2.892	0.430	2.974
45	0.988	2.156	0.938	2.225	0.887	2.296	0.838	2.367	0.788	2.439	0.740	2.512	0.692	2.586	0.644	2.659	0.598	2.733	0.553	2.807
50	1.064	2.103	1.019	2.163	0.973	2.225	0.927	2.287	0.882	2.350	0.836	2.414	0.792	2.479	0.747	2.544	0.703	2.610	0.660	2.675
55	1.129	2.062	1.087	2.116	1.045	2.170	1.003	2.225	0.961	2.281	0.919	2.338	0.877	2.396	0.836	2.454	0.795	2.512	0.754	2.571
60	1.184	2.031	1.145	2.079	1.106	2.127	1.068	2.177	1.029	2.227	0.990	2.278	0.951	2.330	0.913	2.382	0.874	2.434	0.836	2.487
65	1.231	2.006	1.195	2.049	1.160	2.093	1.124	2.138	1.088	2.183	1.052	2.229	1.016	2.276	0.980	2.323	0.944	2.371	0.908	2.419
70	1.272	1.986	1.239	2.026	1.206	2.066	1.172	2.106	1.139	2.148	1.105	2.189	1.072	2.232	1.038	2.275	1.005	2.318	0.971	2.362
75	1.308	1.970	1.277	2.006	1.247	2.043	1.215	2.080	1.184	2.118	1.153	2.156	1.121	2.195	1.090	2.235	1.058	2.275	1.027	2.315
80	1.340	1.957	1.311	1.991	1.283	2.024	1.253	2.059	1.224	2.093	1.195	2.129	1.165	2.165	1.136	2.201	1.106	2.238	1.076	2.275
85	1.369	1.946	1.342	1.977	1.315	2.009	1.287	2.040	1.260	2.073	1.232	2.105	1.205	2.139	1.177	2.172	1.149	2.206	1.121	2.241
90	1.395	1.937	1.369	1.966	1.344	1.995	1.318	2.025	1.292	2.055	1.266	2.085	1.240	2.116	1.213	2.148	1.187	2.179	1.160	2.211
95	1.418	1.929	1.394	1.956	1.370	1.984	1.345	2.012	1.321	2.040	1.296	2.068	1.271	2.097	1.247	2.126	1.222	2.156	1.197	2.186
100	1.439	1.923	1.416	1.948	1.393	1.974	1.371	2.000	1.347	2.026	1.324	2.053	1.301	2.080	1.277	2.108	1.253	2.135	1.229	2.164
150	1.579	1.892	1.564	1.908	1.550	1.924	1.535	1.940	1.519	1.956	1.504	1.972	1.489	1.989	1.474	2.006	1.458	2.023	1.443	2.040
200	1.654	1.885	1.643	1.896	1.632	1.908	1.621	1.919	1.610	1.931	1.599	1.943	1.588	1.955	1.576	1.967	1.565	1.979	1.554	1.991

Note: n = number of observations, k' = number of explanatory variables excluding the constant term.

Source: This table is an extension of the original Durbin–Watson table and is reproduced from N. E. Savin and K. J. White, "The Durbin-Watson Test for Serial Correlation with Extreme Small Samples or Many Regressors," *Econometrica*, vol. 45, November 1977, pp. 1989–96 and as corrected by R. W. Farebrother, *Econometrica*, vol. 48, September 1980, p. 1554. Reprinted by permission of the Econometric Society.

EXAMPLE 1

If $n = 40$ and $k' = 4$, $d_L = 1.285$ and $d_U = 1.721$. If a computed d value is less than 1.285, there is evidence of positive first-order serial correlation; if it is greater than 1.721, there is no evidence of positive first-order serial correlation; but if d lies between the lower and the upper limit, there is inconclusive evidence regarding the presence or absence of positive first-order serial correlation.

	$k' = 11$		$k' = 12$		$k' = 13$		$k' = 14$		$k' = 15$		$k' = 16$		$k' = 17$		$k' = 18$		$k' = 19$		$k' = 20$	
n	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
16	0.098	3.503	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	0.138	3.378	0.087	3.557	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	0.177	3.265	0.123	3.441	0.078	3.603	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	0.220	3.159	0.160	3.335	0.111	3.496	0.070	3.642	—	—	—	—	—	—	—	—	—	—	—	—
20	0.263	3.063	0.200	3.234	0.145	3.395	0.100	3.542	0.063	3.676	—	—	—	—	—	—	—	—	—	—
21	0.307	2.976	0.240	3.141	0.182	3.300	0.132	3.448	0.091	3.583	0.058	3.705	—	—	—	—	—	—	—	—
22	0.349	2.897	0.281	3.057	0.220	3.211	0.166	3.358	0.120	3.495	0.083	3.619	0.052	3.731	—	—	—	—	—	—
23	0.391	2.826	0.322	2.979	0.259	3.128	0.202	3.272	0.153	3.409	0.110	3.535	0.076	3.650	0.048	3.753	—	—	—	—
24	0.431	2.761	0.362	2.908	0.297	3.053	0.239	3.193	0.186	3.327	0.141	3.454	0.101	3.572	0.070	3.678	0.044	3.773	—	—
25	0.470	2.702	0.400	2.844	0.335	2.983	0.275	3.119	0.221	3.251	0.172	3.376	0.130	3.494	0.094	3.604	0.065	3.702	0.041	3.790
26	0.508	2.649	0.438	2.784	0.373	2.919	0.312	3.051	0.256	3.179	0.205	3.303	0.160	3.420	0.120	3.531	0.087	3.632	0.060	3.724
27	0.544	2.600	0.475	2.730	0.409	2.859	0.348	2.987	0.291	3.112	0.238	3.233	0.191	3.349	0.149	3.460	0.112	3.563	0.081	3.658
28	0.578	2.555	0.510	2.680	0.445	2.805	0.383	2.928	0.325	3.050	0.271	3.168	0.222	3.283	0.178	3.392	0.138	3.495	0.104	3.592
29	0.612	2.515	0.544	2.634	0.479	2.755	0.418	2.874	0.359	2.992	0.305	3.107	0.254	3.219	0.208	3.327	0.166	3.431	0.129	3.528
30	0.643	2.477	0.577	2.592	0.512	2.708	0.451	2.823	0.392	2.937	0.337	3.050	0.286	3.160	0.238	3.266	0.195	3.368	0.156	3.465
31	0.674	2.443	0.608	2.553	0.545	2.665	0.484	2.776	0.425	2.887	0.370	2.996	0.317	3.103	0.269	3.208	0.224	3.309	0.183	3.406
32	0.703	2.411	0.638	2.517	0.576	2.625	0.515	2.733	0.457	2.840	0.401	2.946	0.349	3.050	0.299	3.153	0.253	3.252	0.211	3.348
33	0.731	2.382	0.668	2.484	0.606	2.588	0.546	2.692	0.488	2.796	0.432	2.899	0.379	3.000	0.329	3.100	0.283	3.198	0.239	3.293
34	0.758	2.355	0.695	2.454	0.634	2.554	0.575	2.654	0.518	2.754	0.462	2.854	0.409	2.954	0.359	3.051	0.312	3.147	0.267	3.240
35	0.783	2.330	0.722	2.425	0.662	2.521	0.604	2.619	0.547	2.716	0.492	2.813	0.439	2.910	0.388	3.005	0.340	3.099	0.295	3.190
36	0.808	2.306	0.748	2.398	0.689	2.492	0.631	2.586	0.575	2.680	0.520	2.774	0.467	2.868	0.417	2.961	0.369	3.053	0.323	3.142
37	0.831	2.285	0.772	2.374	0.714	2.464	0.657	2.555	0.602	2.646	0.548	2.738	0.495	2.829	0.445	2.920	0.397	3.009	0.351	3.097
38	0.854	2.265	0.796	2.351	0.739	2.438	0.683	2.526	0.628	2.614	0.575	2.703	0.522	2.792	0.472	2.880	0.424	2.968	0.378	3.054
39	0.875	2.246	0.819	2.329	0.763	2.413	0.707	2.499	0.653	2.585	0.600	2.671	0.549	2.757	0.499	2.843	0.451	2.929	0.404	3.013
40	0.896	2.228	0.840	2.309	0.785	2.391	0.731	2.473	0.678	2.557	0.626	2.641	0.575	2.724	0.525	2.808	0.477	2.892	0.430	2.974
45	0.988	2.156	0.938	2.225	0.887	2.296	0.838	2.367	0.788	2.439	0.740	2.512	0.692	2.586	0.644	2.659	0.598	2.733	0.553	2.807
50	1.064	2.103	1.019	2.163	0.973	2.225	0.927	2.287	0.882	2.350	0.836	2.414	0.792	2.479	0.747	2.544	0.703	2.610	0.660	2.675
55	1.129	2.062	1.087	2.116	1.045	2.170	1.003	2.225	0.961	2.281	0.919	2.338	0.877	2.396	0.836	2.454	0.795	2.512	0.754	2.571
60	1.184	2.031	1.145	2.079	1.106	2.127	1.068	2.177	1.029	2.227	0.990	2.278	0.951	2.330	0.913	2.382	0.874	2.434	0.836	2.487
65	1.231	2.006	1.195	2.049	1.160	2.093	1.124	2.138	1.088	2.183	1.052	2.229	1.016	2.276	0.980	2.323	0.944	2.371	0.908	2.419
70	1.272	1.986	1.239	2.026	1.206	2.066	1.172	2.106	1.139	2.148	1.105	2.189	1.072	2.232	1.038	2.275	1.005	2.318	0.971	2.362
75	1.308	1.970	1.277	2.006	1.247	2.043	1.215	2.080	1.184	2.118	1.153	2.156	1.121	2.195	1.090	2.235	1.058	2.275	1.027	2.315
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150	1.579	1.892	1.564	1.908	1.550	1.924	1.535	1.940	1.519	1.956	1.504	1.972	1.489	1.989	1.474	2.006	1.458	2.023	1.443	2.040
200	1.654	1.885	1.643	1.896	1.632	1.908	1.621	1.919	1.610	1.931	1.599	1.943	1.588	1.955	1.576	1.967	1.565	1.979	1.554	1.991

Note: n = number of observations, k' = number of explanatory variables excluding the constant term.

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Statistiska institutionen

Stockholms universitet

Rättningsblad

Datum: 27/4-2017

Sal: Brunnsvikssalen

Tenta: Regressionsanalys

Kurs: Ekonometri

ANONYMKOD: EKR-0014

☑ Jag godkänner att min tenta får läggas ut anonymt på hemsidan som studentsvar.

OBS! SKRIV ÄVEN PÅ BAKSIDAN AV SKRIVBLADEN

Markera besvarade uppgifter med kryss

1	2	3	4	5	6	7	8	9	Antal inl. blad
X	X	X	X	X					5
Lär.ant. 22	25	17	12	15					

POÄNG	**BETYG**	**Lärarens sign.**
91	A	

SU, STATISTIK

Skrivsal: UG **Anonymkod:** EKR-0014 **Blad nr:** -1-

Exercise 1.

- y - life time (in min.) of a cutting tool;
- X - speed (rounds / minute); A, B - types of tools.
- $y_i = \beta_1 + \beta_2 \cdot X_i + \beta_3 \cdot D_i + u_i$, $n = 20$

$$D = \begin{cases} 1 & \text{if } B \\ 0 & \text{if } A \end{cases}$$

a) $R^2 = \frac{ESS}{TSS} = \frac{1418,034}{1545,089} \cong 0,90 \approx (90\%)$

$$\bar{R}^2 = 1 - \frac{RSS/(n-k)}{TSS/(n-1)} = 1 - \frac{154,055/17}{1545,089/19} =$$

$$= 1 - \frac{9,2385}{81,3994} \cong 1 - 0,1114 \approx 0,889 \approx (88,9\%)$$

Answer: $R^2 \cong 90\%$, $\bar{R}^2 \cong 88,9\%$

Ok

b) We test this with a general F-test.

- Hypothesis:

H_0: $\beta_2 = \beta_3 = 0$

H_a: at least one of the β's $\neq 0$.

- Test statistics: $F = \frac{ESS/(k-1)}{RSS/(n-k)} \overset{H_0}{\sim} F(k-1, n-k; \alpha)$.

- $F_{crit.\,(2,17)}^{(0,01)} = 6,11$

- Decision: If $F_{obs} > F_{crit}$ we reject H_0 in favor of H_a

- $F_{obs} = \frac{1418,034/2}{154,055/17} = \frac{709,017}{9,2385} \cong 76,4$

→

Ex 1. cont

Answer: $F_{obs} = 76,7 > F_{crit} = 6,11$, therefore we reject H_0 at 1% sign level, we've got evidence that at least one of the expl. variables should be in the model.

OK

d) The model: $Y_i = \beta_1 + \beta_2 \cdot X_i + \beta_3 \cdot D_i + u_i$

Our model includes the intercept term and a dummy variable. It means that we treat β_1 not only as intercept, but also as the reference base (at least theoretically). Therefore β_1 represents here the (average) effective life time of a type A tool, then β_3 is a differential effect for a type B tool (compared with a type A).

$$\hat{Y}_i = 36,986 - 0,024 \cdot \hat{X} + 15,004 \cdot D_i$$

→ β_1 = (average) effective life time of a type A tool. = 36,986

→ β_3 = indicates that the (average) effective life time of a type B tool differs from that of a type A by + 15,004 and is equal to 36,986 + 15,004 = 51,99; holding X constant (essentially when X = 0).

Answer: The expectation life time of a type B tool is 51,99 (mins).

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Cont. Ex. 1

d) C.i. for β_3 at 95%:

$$\hat{\beta}_3 \pm t^{(n-k)}_{\alpha/2} \cdot SE(\hat{\beta}_3).$$

- $\hat{\beta}_3 = 15{,}004$
- $se(\hat{\beta}_3) = 1{,}36$
- $t^{(17)}_{0{,}025} = 2{,}11$

$$\Rightarrow 15{,}004 \pm 2{,}11 \cdot 1{,}36 = $$
$$= 15{,}004 \pm 2{,}8696$$

Answer: $I^{\alpha=0{,}05}_{\beta_3} = [12{,}13 ; 17{,}87]$ OK

e) (1): $y_i = \beta_1 + \beta_2 \cdot X_i + \beta_3 \cdot D_i + u_i$

(2): $y_i = \beta_1' + \beta_2' X_i + \beta_3' \cdot D_i + \underline{\beta_4 X_i \cdot D_i} + \varepsilon_i$.

The effective (expected) life time ($\hat{y}$) depends on (if $D = 1$, type B tool).

→ In model (1): $\hat{y} = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_i + \hat{\beta}_3 =$

$= (\hat{\beta}_1 + \hat{\beta}_3) + \hat{\beta}_2 \cdot X_i$ → it will be determined by the speed, ~~whether~~ it's the speed of a type A or a type B tool is unspecified.

→ In model (2): $\hat{y} = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_i + \hat{\beta}_3 + \hat{\beta}_4 \cdot X_i$

$= (\hat{\beta}_1 + \hat{\beta}_3) + (\hat{\beta}_2 + \hat{\beta}_4) \cdot X_i$

Here, the expected effective life time $\hat{y}$ for a type B tool will be influenced by the speed of a type B tool and will be "corrected" by $\hat{\beta}_4$ (differential effect of the speed of a type B tool). In this respect the model (2) is richer

Ok

Ex cont.

f) Test stat: $F = \frac{(R^2_{new} - R^2_{old}) / m}{(1 - R^2_{new}) / (n - k)}$ (m = number of new regs; k = number of param. of the new model)

$\overset{H_0}{\sim} F^{(\alpha)}(m; n-k)$.

- **H0**: the term X_iD_i does not contribute significantly to the model
- **Ha**: it does contribute significantly to the model

Note: we can use R^2 in the test as both the two model have the same form (funct).

- $\underline{F_{cr}^{(0,05)}(1; 16)} = 4,49$ (critical)
- **Decision**: if $F_{obs} > F_{crit} \rightarrow$ Reject H0.
- $R^2_{new} = \frac{ESS_{new}}{TSS} = \frac{1434,112}{1575,089} \cong 0,91$
- $F_{obs} = \frac{(0,91 - 0,90)/1}{(1 - 0,91)/16} = \frac{0,01}{0,005625} \cong 1,78$

Answer: as $F_{obs} \cong 1,78 < F_{crit} = 4,49$ we can not reject the H0 and therefore we can not conclude that the new model is better (we did not get enough evidence that X_iD_i should be included) at $\alpha = 0,05$.

OK

Exercise 2

- $Q = \beta_1 \cdot L^{\beta_2} \cdot K^{\beta_3}$, L = labour input, K = cap stock

$y = \ln(Q)$ as dependent in a linear RM.

a) $\ln(Q) = \ln\beta_1 + \beta_2 \cdot \ln(L) + \beta_3 \cdot \ln(K) + u_i$

OK

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Skrivsal: UC **Anonymkod:** EKR-0014 **Blad nr:** - 3 -

Ex. 2, cont.

b) $\beta_3 = 1 - \beta_2$

$$\ln(Q) = \ln\beta_1 + \beta_2 \ln(L) + (1-\beta_2)\cdot \ln K =$$
$$= \ln\beta_1 + \beta_2 \ln(L) + \ln K - \beta_2 \cdot \ln K =$$
$$= \ln\beta_1 + \ln K + \beta_2(\ln(L) - \ln(K))$$

$$\ln(Q) - \ln(K) = \ln\beta_1 + \beta_2(\ln(L) - \ln(K))$$

$$\Rightarrow \ln\left(\frac{Q}{K}\right) = \ln\beta_1 + \beta_2 \ln\left(\frac{L}{K}\right) =$$
$$= \beta_1' + \beta_2 \cdot \ln\left(\frac{L}{K}\right), \ (+u_i)$$

where $\beta_1' = \ln\beta_1$

OK

c) $n = 33$

(1) $\ln Q = \ln\beta_1 + \beta_2 \cdot \ln L + \beta_3 \cdot \ln K + u_i$

(2) $\ln\left(\frac{Q}{K}\right) = \ln\beta_1 + \beta_2 \cdot \ln\left(\frac{L}{K}\right) + u_i$

Since we have used 1 restriction, namely $\beta_3 = 1 - \beta_2$, to obtain the model (2), I suggest to use F-restrictedtest as a test statistics at $\alpha = 0{,}05$.

- H_0: the restricted model (2) is better

H_a: the unrestricted model (1) is better

- Test stat: $F = \frac{(ESS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)} \overset{H_0}{\sim} F^{0{,}05}_{(1,30)}$

- $F_{crit}{}^{(0{,}05)}_{(1,30)} = 4{,}17$

OK

Ex. 2, cont

- reject H_0 if $F_{obs} > F_{crit} = 4{,}17$ at $\alpha = 0{,}05$.

d) One potential advantage, from a modelling point of view, is that model in (b) is built on ratios $\frac{Q}{K}$ and $\frac{L}{K}$, which are used in economics and designate

$\frac{L}{K}$ = labor/capital ratio, and

$\frac{Q}{K}$ = production/capital ratio. From this model we can directly estimate the elasticity of $\frac{Q}{K}$ measured with regard to a unit (%) change in $\frac{L}{K}$. OK

e) $\ln Q = \ln \beta_1 + \beta_2 \cdot \ln(L) + \beta_3 \cdot \ln(K) + u$

- $\hat{\rho} = 0{,}6$, $\alpha = 0{,}05$.
- 1899 - 1922, $n = 24$

→ In order to draw the conclusion about the presence of positive autocorrelation we have to find out the Durbin Watson d statistic. We can do it on the basis of $\hat{\rho}$, knowing that:

$$d \approx 2(1-\hat{\rho}) \approx 2(1-0{,}6) = 0{,}8$$

→ Critical values for d at $\alpha = 0{,}05$ $\left(\begin{matrix} k = 2 \\ n = 24 \end{matrix}\right)$

$d_L = 1{,}188$ $d_U = 1{,}546$

→ $d = 0{,}8 < d_L = 1{,}188$, which does confirm that we have evidence of a positive autocorrelation.

125 OK

SU, STATISTIK

EKR-0014

Skrivsal: UC **Anonymkod:** **Blad nr:** -4-

Exercise 3

- $n = 108$ (single households),
  y = rent, x = income
- $y = \beta_1 + \beta_2 \cdot x + u_i$ (simple linear).

a) $\hat{u}_i^2 = \beta_1 + \beta_2 \cdot x + \beta_3 \cdot x^2 + v_i$

b) $n = 108$, $R^2_{(aux)} = 0,082134$.

- Test statistic: $n \cdot R^2 \overset{H_0}{\sim} \chi^2(2)$
- $n \cdot R^2 = 108 \cdot 0,082134 \approx 8,87$
- $\chi^{2\,0,05}_{obs}(2) = 5,99$

Answer: $\chi^2_{obs} = 8,87 > \chi^2_{crit} = 5,99$ therefore we can reject H_0 at 5% sign. level. We've got evidence of heteroskedasticity in the model.

c) $V(u_i) = \sigma^2 \cdot x_i$,

Mathematically:

$$V(a \cdot u_i) = a^2 \cdot V(u_i) = a^2 \cdot \sigma^2 x_i$$

We seek: $a^2 \cdot \sigma^2 \cdot x_i = \sigma^2$

$$a^2 = \frac{1}{x_i} \rightarrow a = \frac{1}{\sqrt{x_i}}$$, that's why to correct for the heteroskedasticity in our model we can transform our data by dividing it by $\sqrt{x_i}$:

$$\frac{y_i}{\sqrt{x_i}} = \frac{\beta_1}{\sqrt{x_i}} + \frac{\beta_2 \cdot x_i}{\sqrt{x_i}} + \frac{u_i}{\sqrt{x_i}} = \frac{\beta_1}{\sqrt{x_i}} + \beta_2 \sqrt{x_i} + \frac{u_i}{\sqrt{x_i}}$$

17

("no-intercept" model).

Exercise 4

- true model: $y_i = \beta_1 + \beta_2 X_i + u_i$

  using: $(y_i - c_1) = \beta_1 + \beta (X_i - c_2) + u_i$

a) we are using the formulas:

$$\boxed{\begin{aligned}\hat\beta_1 &= \bar{y} - \hat\beta_2 \bar{X}\\ \hat\beta_2 &= \frac{\sum (X_i - \bar{X})(y_i - \bar{y})}{\sum (X_i - \bar{X})^2}\end{aligned}}$$

$\boxed{c_1 \text{ and } c_2 \text{ are known constants!}}$

- $\underbrace{(y_i - c_1)} = \beta_1 + \beta_2 \cdot (X_i - c_2) + u_i$
- $y^* = \beta_1 + \beta_2 X^* + u_i$
- we use the fact that

  $\bar{X}^* = \overline{X_i - c_2} = \bar{X}_i - c_2$, ($c_2$ is a constant),

  $\bar{y}^* = \overline{y_i - c_1} = \bar{y} - c_1$

- $$\boxed{\hat\beta_2} = \frac{\sum (X_i^* - \bar{X}^*)(y_i^* - \bar{y}^*)}{\sum (X_i^* - \bar{X}^*)} = \sum($$

$$= \frac{\sum (X_i - c_2 - (\bar{X} - c_2))(y_i - c_1 - (\bar{y} - c_1))}{\sum (X_i - c_2 - (\bar{X} - c_2))^2} =$$

$$= \frac{\sum (X_i - \bar{X})(y_i - \bar{y})}{\sum (X_i - \bar{X})^2} = \hat{} \quad = \text{OLS estimator for } \hat\beta_2 \ ($$

Ok

- $$\boxed{\hat\beta_1} = \bar{y}^* - \hat\beta_2 \bar{X}^* = \bar{y} - c_1 - \hat\beta_2 (\bar{X} - c_2) =$$

  $$= \bar{y} - \hat\beta (\bar{X} - c_2) \underline{- c_1} \quad - \text{OLS estimator for } \hat\beta_1$$

Ok

b) $E(\hat\beta_2) = \beta_2$ – an <u>unbiased</u> estimator,

$E(\hat\beta_1) \neq \beta_1$ – <u>biased</u>, basically it's seen

from $y_i = (\beta + c) + \beta (X \quad c) + u_i$ * continued on the last page

SU, STATISTIK

Skrivsal: Ul **Anonymkod:** EKR-0014 **Blad nr:** -5-

Exercise 5

a) If X is divided by 2, then:

$$\bullet\ \hat{\beta}_2^* = \frac{\sum\left(\frac{X_i}{2}-\frac{\bar{X}}{2}\right)\left(Y_i-\bar{Y}\right)}{\sum\left(\frac{X_i}{2}-\frac{\bar{X}}{2}\right)^2} =$$

$$= \frac{\frac{1}{2}\sum\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)}{\frac{1}{4}\sum\left(X_i-\bar{X}\right)^2} = 2\cdot\hat{\beta}_2$$

$$\bullet\ \hat{\beta}_1^* = \bar{Y} - \hat{\beta}_2^*\cdot\frac{\bar{X}}{2} = \bar{Y} - 2\hat{\beta}_2\cdot\frac{\bar{X}}{2} = \bar{Y} - \hat{\beta}_2\cdot\bar{X} = \hat{\beta}_1$$

FALSE. Only $\hat{\beta}_1$ is unchanged Ok

b) $Y_i = \beta_1 + \beta_2 X_i + u_i$, $Y^* = Y_i + \varepsilon_i$ (ε_i is uncorrelated w/ Y^* and Y_i, $E(\varepsilon_i) = 0$ usual assumpt.)

$Y^* - \varepsilon_i = \beta_1 + \beta_2 X_i + u_i$

$Y^* = \beta_1 + \beta_2 \cdot X_i + \left(u_i + \varepsilon_i\right)$

$\Rightarrow \operatorname{Cov}(X_i, u_i) = 0$, $\operatorname{Cov}(u_i + \varepsilon_i) = 0$.

FALSE. X and error term are uncorrelat Ok

c) FALSE. the test is used to check Ok
if there is autocorrelation. So, if the disturbance variables are autocorrelated it is used to detect it. One remark here

It's used only to detect autoco- rr. only the presence of the first-ord Markov correlation scheme, i.e. (AR1): $u_t = \rho \cdot u_{t-1} + \varepsilon_t$

d) **FALSE**. COVARIANCE can be any number. It is coefficient of correlation r (ρ) that lies in the interval $[-1, 1]$ which is determined (if I remember it correctly) as $\rho = \frac{cov(X,Y)}{SE_X \cdot SE_Y}$. OK

e) **TRUE** ~~(partially)~~ It can be one of the reasons of heteroscedasticity (i.e. $V(u_i) = \sigma^2 X_i^2$, though it's not the only reason. OK

f) **FALSE** ✓ The runs test doesn't rely on the assumption of normality. It's the Durbin Watson d test (another test to detect autocorrelation) that relies on it. /2

* Exercise 4, cont.

$E(\tilde{\beta}_1) = E(\bar{Y} - \hat{\beta}_2 \cdot \bar{X} + \hat{\beta}_2 \cdot C_2 - C_1) =$

$= E(\bar{Y} - \hat{\beta}_2 \cdot \bar{X}) + E(\hat{\beta}_2 \cdot C_2 - C_1) =$

$= E(\hat{\beta}_1) + E(\hat{\beta}_2 \cdot C_2) - E(C_1) =$

$= \beta_1 + \underbrace{C_2 \cdot \beta_2 - C_1} \longrightarrow$ bias OK /1

$E(\hat{\beta}^1)$ β [illegible] not unbiased

Statistiska institutionen

Stockholms universitet

Rättningsblad

Datum: 27/4-2017

Sal: ~~Brunnsvikssa~~len UG

Tenta: Regressionsanalys

Kurs: Ekonometri

ANONYMKOD: EKR-0025

☒ Jag godkänner att min tenta får läggas ut anonymt på hemsidan som studentsvar.

OBS! SKRIV ÄVEN PÅ BAKSIDAN AV SKRIVBLADEN

Markera besvarade uppgifter med kryss

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Antal inl. blad**
X	X	X	X	X					6
Lär.ant. 23	20	20	10	16					

POÄNG	**BETYG**	**Lärarens sign.**
89	A	

SU, STATISTIK

Skrivsal: UG **Anonymkod:** EKR-0025 **Blad nr:** ①

1) $Y_i = \beta_1 + \beta_2 X_i + \beta_3 D_i + u_i$

$n = 20$

$k = 3$

a)

Coeff. of determination $R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{157,055}{1575,089} = 0,900$

Adjusted $\bar{R}^2 = 1 - \frac{RSS/(n-k)}{TSS/(n-1)} = 1 - \frac{157,055/(20-3)}{1575,087/(20-1)}$

$ESS = 1418,034$

$RSS = 157,055$

$TSS = 1575,089$

$= 0,889$

$R^2 = 0,900$ $\bar{R}^2 = 0,889$ OK

b) $H_0: \beta_2 = \beta_3 = 0$ (none of the variables influence Y_i)

H_A: At least one of β_2 or β_3 is not 0

General F-test for H_0: $F = \frac{ESS/(k-1)}{RSS/(n-k)} = \frac{1418,034/2}{157,055/17}$

$\Rightarrow F = 76,746$

Critical F from table

$F_{0,01}(2,17) = 6,11$

$F > F_{crit} \Rightarrow$ H_0 is rejected and at least one of β_2 or β_3 should be included in the model

OK

c)

Type A $E(Y_i \mid D=0) = \beta_1 + \beta_2 x_i$

Type B $E(Y_i \mid D=1) = \beta_1 + \beta_2 X_i + \beta_3$

β_3 is the difference in the expected value of Y for a given X i.e the difference in life time between A and B for a given speed

OK

d) 95% confidence interval

$$\frac{\hat{\beta}_i - \beta_i}{se(\hat{\beta}_i)} \sim t(n-k)$$

Under $H_0 : \beta_i = 0$

$$\hat{\beta}_i \pm t(n-k) \cdot se(\hat{\beta}_i)$$

$$\hat{\beta}_3 \pm t_{0,05}(17) \cdot se(\hat{\beta}_3)$$

$$15{,}004 \pm \underbrace{2{,}11 \cdot 1{,}36}_{2{,}8696}$$

$12{,}13 - 17{,}87$ 95% conf. interval for β_3

OK

SU, STATISTIK

Skrivsal: U-G **Anonymkod:** EKR-0025 **Blad nr:** ②

Uppgift 1

e) $Y_i = \beta_1' + \beta_2' X_i + \beta_3' D_i + \beta_4' X_i D_i + u_i$

$k=4$

$n=20$

This model also includes an interaction term between X_i and D_i, the speed and the type. Therefore this model can account for a possible interaction between speed and type that is multiplicative ?, $\beta_4 X_i D_i$.

f) Test if the new model is sign. better

H_0: The new model is not sign. better than the old.

H_A: the new -11- is better.

$$F = \frac{(ESS_{new} - ESS_{old}) / 1}{RSS_{new} / (n-4)} = \frac{1434{,}112 - 1418{,}034}{140{,}977 / 16}$$

$ESS_{new} = 1434{,}112$

$ESS_{old} = 1418{,}034$

$RSS_{new} = 140{,}977$

$\Rightarrow$

$F = 1{,}82$

$F_{0,05}(1,16) = 4{,}49$ (table)

$F < F_{crit}$ so H_0 is not rejected and the interaction term should not be included in the model

Ok /23

SU, STATISTIK

Skrivsal: UG   Anonymkod: EKR-0025   Blad nr: ③

Uppgift 2 $Q = \beta_1 L^{\beta_2} K^{\beta_3}$

L = labour input

K = capital stock

a) $Y = \ln Q = \underbrace{\ln \beta_1}_{\alpha_1} + \beta_2 \ln L + \beta_3 \ln K + u_i$ Ok

b) $\beta_3 = 1 - \beta_2 \Rightarrow Y = \alpha_1 + \beta_2 \ln L + (1-\beta_2) \ln K + u_i$

$\Rightarrow Y = \alpha_1 + \beta_2 (\ln L - \ln K) + \ln K + u_i$

$Y - \ln K = \alpha_1 + \beta_2 (\ln L - \ln K) + u_i$ Ok

c) To test the assumption that $\beta_3 = 1 - \beta_2$ we set up

$H_0: \beta_2 + \beta_3 = 1$

$H_A: \beta_2 + \beta_3 \neq 1$

$$\frac{\hat{\beta}_i - \beta_i}{se(\hat{\beta}_i)} \sim t(n-k)$$

Under H_0

$$\frac{\hat{\beta}_2 + \hat{\beta}_3 - 1}{se(\hat{\beta}_2 + \hat{\beta}_3)} \sim t(33-3=30)$$

If $t < -2{,}042$ or $t > 2{,}042$ we reject H_0 and do not implement $\beta_2 + \beta_3 = 1$ in the model

Inte samt att $E(\hat{\beta}_2 + \hat{\beta}_3) = 1$ under H_0.

d) If $\beta_3 - 1 - \beta_2$ and we have both β_2 and β_3 in the model as we have in a) we will might have a problem with collinearity as β_3 is a linear function of β_2. Model b) might solve this problem.

e) I don't remember how to calculate d from $\hat{\rho}$ but if I could calculate d I would compare to d_L in Durbin Watson table

$k' = 2$ $n = 24$ (1899-1922 yearly)

Change to 5% significance as table is missing

$d_L = 1{,}188$ $d_u = 1{,}546$

If $d \leq d_L$ there is positive autocorrelation

/20

SU, STATISTIK

Skrivsal: UG **Anonymkod:** EKR-0025 **Blad nr:** ④

Uppgift 3. $Y_i = \beta_1 + \beta_2 X + u_i$ Y: Rent
X = income

a) White's auxilliary regression

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 X + \alpha_3 X^2 + \varepsilon_i$$

OK

b) H_0: Homoscedaucity $\alpha_2 = \alpha_3 = 0$

H_A: Heteroscedaucity

$$nR^2 \sim \chi^2(2)$$ approx. under H_0

$n = 108$ $R^2 = 0{,}082134 \Rightarrow n \cdot R^2 = 8{,}87$

$\chi^2_{0,05}(2) = 5{,}991$

Calculated $\chi^2 = nR^2 > \chi^2_{0,05}(2)$ H₀ is rejected

There is heteroscedaucity

OK

3.c $V(u_i) = \sigma^2 X_i$ $\quad$ $Y = \beta_1 + \beta_2 X_i + u_i$

$$V\left(\frac{u_i}{\sqrt{X_i}}\right) = \frac{1}{X_i} \cdot V(u_i) = \frac{1}{X_i} \cdot \sigma^2 X_i = \sigma^2$$

homoscedasticity

X_i is not random

The model can be transformed to obtain homosc.

$$\frac{Y_i}{\sqrt{X_i}} = \beta_1 \cdot \frac{1}{\sqrt{X_i}} + \beta_2 \sqrt{X_i} + \frac{u_i}{\sqrt{X_i}}$$

OK

/20

SU, STATISTIK

Skrivsal: UG **Anonymkod:** EKR-0025 **Blad nr:** ⑤

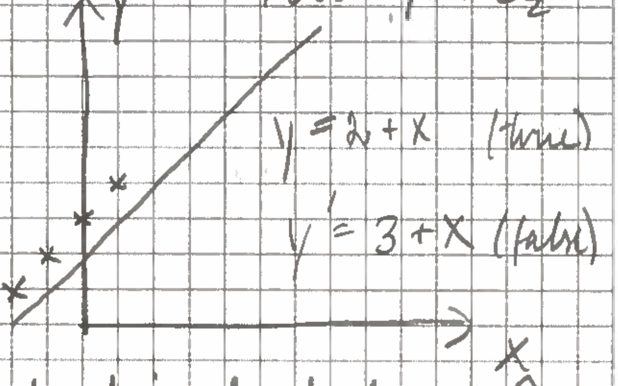
Uppgift 4 $Y_i = \beta_1 + \beta_2 X_i + u_i$ (True)

$(Y_i - c_1) = \beta_1 + \beta_2 (X_i - c_2) + u_i$ (False)

a) $Y_i = \beta_1 + c_1 + \beta_2 (X_i - c_2)$

Test $c_1 = 1$ $c_2 = 2$

Y-1	Y	X	X-2
1	2	0	-2
2	3	1	-1
3	4	2	0
4	5	3	1



Guessing that $\hat{\beta}_1$ will be biased but not $\hat{\beta}_2$.

$E(\hat{\beta}_i) = \beta_i$ unbiased

$$\hat{\beta}_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} =$$

$$\frac{\sum (X_i - \cancel{c_2} - \bar{X} + \cancel{c_2})(Y_i - \cancel{c_1} - \bar{Y} + \cancel{c_1})}{\sum (X_i - \cancel{c_2} - \bar{X} + \cancel{c_2})^2} = \beta_2$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = \bar{Y} - c_1 - \hat{\beta}_2 (\bar{X} - c_2) \neq \beta_1$$

$\hat{\beta}_1$ is biased and $\hat{\beta}_2$ is unbiased

Show that!

1/10

SU, STATISTIK

Skrivsal: UG **Anonymkod:** EKR-0025 **Blad nr:** 6

Uppgift 5

a) $Y = \beta_1 + \beta_2 X + u_i \qquad X^* = \frac{X}{2}$

$\hat{\beta}_1^* = \hat{\beta}_1 \qquad \hat{\beta}_2^* = 2\hat{\beta}_2$

False, only $\hat{\beta}_2$ is changed OK

b) False Y_i is the dependent variable, a measurement error in X_i can introduce $\operatorname{cov}(X_i, u_i) \neq 0$ OK

c) False. The DB test is used to detect autocorr. in disturbance terms OK

d) False, the coeff. of correlation is $-1 < r < 1$ not the cov. OK

e) True. When $V(u_i) = \sigma_i^2$ and the variance changes with i.e x_i there is heteroscedasticity OK/3

f) True. The number of runs $R \sim f$ under H_0: R is random when $N_1 > 10$ and $N_2 > 10$ OK

//6