Generalized Quantifiers:
Linguistics meets Model Theory

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0.1 Thirty years of Generalized Quantifiers

It is now more than 30 years since the first serious applications of Generalized Quantifier (GQ) theory to natural language semantics were made: Barwise and Cooper (1981), Keenan and Stavi (1986), Higginbotham and May (1981). Richard Montague had in effect interpreted English NPs as (type $\langle 1\rangle$) generalized quantifiers (see Montague (1974)),[1] but without referring to GQs in logic, where they had been introduced by Mostowski (1957) and, in final form, Lindström (1966). Logicians were interested in the properties of *logics* obtained by piecemeal additions to first-order logic (FO) by adding quantifiers like 'there exist uncoutably many', but they made no connection to natural language.[2] Montague Grammar and related approaches had made clear the need for higher-type objects in natural language semantics. What Barwise, Cooper, and the others noticed was that generalized quantifiers are the natural interpretations not only of noun phrases but in particular of *determiners*.[3]

This was no small insight, even if it may now seem obvious. Logicians had, without intending to, made available model-theoretic objects suitable for interpreting English definite and indefinite articles, the aristotelian `all, no, some`, proportional Dets like `most, at least half, 10 percent of the, less than two-thirds of the`, numerical Dets such as `at least five, no more than ten, between six and nine, finitely many, an odd number of`, definite Dets like `the, the twelve`, possessives like `Mary's, few students', two of every professor's`, exception Dets like `no...but John, every...except Mary`, and boolean combinations of all of the above. *All* of these can—if one wants!—be interpreted extensionally as the same type of second-order

[1] In this paper I use the classical terminology, but the reader is free to substitute everywhere 'DP' for 'NP', and 'NP' for 'N'.

[2] Long before them, however, Frege had introduced quantifiers as second-order relations, and he did consider these for interpreting Dets like `all` and `some`, although his main interest was quantification in logic, where the unary $\forall$ was enough. But it was Montague's idea of interpreting NPs as type $\langle 1\rangle$ generalized quantifiers that marked the real starting-point for compositional model-theoretic semantics for natural languages.

[3] All languages appear to have rich means of expressing quantification. Some languages make scarce or no use of Dets but rely instead on 'A quantification', where 'A' stands for adverbs, auxiliaries, affixes, and argument structure adjusters; see Bach *et al.* (1995) for this terminology and many examples. But whatever words or morphemes are used, they can be interpreted, just like Dets, as type $\langle 1, 1\rangle$ quantifiers. It has been claimed that some languages lack NPs, or in general phrases interpretable as type $\langle 1\rangle$ quantifiers. But the point here is that $\langle 1, 1\rangle$ is the basic type of quantification in all natural languages; see Peters and Westerståhl (2006), ch. 0, for discussion.

objects, namely, (on each universe of discourse) *binary relations between sets*. Given the richness of this productive but seemingly heterogeneous class of expressions, a uniform interpretation scheme was a huge step. Further, the tools of logical GQ theory could be brought to bear on putative Det interpretations, which turned out to be a subclass of the class of all type $\langle 1,1\rangle$ quantifiers with special traits. The three pioneer papers mentioned above offered numerous cases of novel description, and sometimes explanation, of characteristic features of language in terms of model-theoretic properties of the quantifiers involved.

This development has continued, and still goes on. Many of the early results have reached the status of established facts in (most of) the linguistic community, and generalized quantifiers are by now standard items in the semanticist's toolbox. In the following sections I will, after a few preliminaries, indicate some of the most important achievements of GQ theory applied to natural language. Each will be presented in this format: first, a feature of language will be identified, and then we will see what GQ theory has to offer. In most cases I will only be able to outline the main issues, and give references to more detailed accounts.

0.2 Definitions, examples, terminology

1 in 'type $\langle 1,1\rangle$' stands for 1-ary relation, i.e. set, so a type $\langle 1,1\rangle$ quantifier (from now on I will often drop the word 'generalized') is (on each universe) a relation between two sets, a type $\langle 1,1,1,1\rangle$ quantifier a relation between four sets, a type $\langle 1,2\rangle$ quantifier a relation between a set and a binary relation. In general, type $\langle n_1,\ldots,n_k\rangle$ signifies a relation between relations $R_1,\ldots,R_k$, where R_i is n_i-ary.

In model theory, a relation is always over a *universe*, which can be any non-empty set M. In a linguistic context we can think of M as a universe of discourse. So by definition, a *quantifier* Q of type $\langle n_1,\ldots,n_k\rangle$ is a function associating with each M a *quantifier* Q_M *on* M of that type, i.e. a k-ary relation between relations over M as above. For $R_i \subseteq M^{n_i}$,

(1) $\quad Q_M(R_1,\ldots,R_k)$

means that the relation Q_M holds for the arguments $R_1,\ldots,R_k$.[4]

[4] We may define instead, as in Lindström (1966), Q as a *class of models* of type $\langle n_1,\ldots,n_k\rangle$. This is really just a notational variant; we have

$$(M,R_1,\ldots,R_k)\in Q \iff Q_M(R_1,\ldots,R_k)$$

Q_M is often called a *local* quantifier (and Q a *global* one). In some applications, the universe can be held fixed, so a local perspective is adequate. But in others, one needs to know how the *same* quantifier behaves in *different* universes. It is important to keep in mind that quantifiers are essentially global objects.[5]

As noted, the most important type in natural language contexts is $\langle 1, 1\rangle$. Here are the interpretations of some of the Dets mentioned in the previous section: For all M and all $A, B \subseteq M$,

(2) $\mathit{all}_M(A, B) \Leftrightarrow A \subseteq B$
$\mathit{some}_M(A, B) \Leftrightarrow A \cap B \neq \emptyset$
$\mathit{no}_M(A, B) \Leftrightarrow A \cap B = \emptyset$
$\mathit{most}_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$ ($|X|$ is the cardinality of X)
$\mathit{less\ than\ two\text{-}thirds\ of\ the}_M(A, B) \Leftrightarrow |A \cap B| < 2/3 \cdot |A|$
$\mathit{at\ least\ five}_M(A, B) \Leftrightarrow |A \cap B| \geq 5$
$\mathit{between\ six\ and\ nine}_M(A, B) \Leftrightarrow 6 \leq |A \cap B| \leq 9$
$\mathit{finitely\ many}_M(A, B) \Leftrightarrow A \cap B$ is finite
$\mathit{an\ odd\ number\ of}_M(A, B) \Leftrightarrow |A \cap B|$ is odd
$\mathit{the\ twelve}_M(A, B) \Leftrightarrow |A| = 12$ and $A \subseteq B$
$\mathit{some\ students'}_M(A,B) \Leftrightarrow \mathit{student} \cap \{a : A \cap \{b : \mathit{has}(a, b)\} \subseteq B\} \neq \emptyset$
$\mathit{every\ldots\ except\ Mary}_M(A, B) \Leftrightarrow A - B = \{m\}$

The notation used in (1) and (2) is set-theoretic. Linguists often prefer lambda notation, from the *simply typed lambda calculus*. This is a *functional* framework, where everything, except primitive objects like individuals (type e) and truth values (type t), is a function. Sets of individuals are (characteristic) functions from individuals to truth values; thus of type $\langle e, t\rangle$. In general, $\langle \sigma, \tau\rangle$ is the type of functions from objects of type σ to objects of type τ. Binary relations are of type $\langle e, \langle e, t\rangle\rangle$, type $\langle 1\rangle$ quantifiers now get the type $\langle\langle e.t\rangle, t\rangle$, and type $\langle 1, 1\rangle$ quantifiers are of type $\langle\langle e, t\rangle\langle\langle e.t\rangle, t\rangle\rangle$.

If *function application* is seen as the major operation that composes meanings (as Frege perhaps thought and Montague showed that one could assume in many, though not all, cases), then the functional notation serves a *compositional* account well. For example, while

[5] Couldn't we simply let a quantifier Q be a second-order relation over the class V of *all* sets, and then define Q_M as the restriction of Q to relations over M? This works for some quantifiers but not others (it works for EXT quantifiers; see section 0.4). For example, the standard universal quantifier $\forall$ would then denote $\{V\}$, but the restriction to a set M is $\emptyset$, rather than $\{M\}$ as desired.

(3) Mary likes Sue.

is simply rendered as

$$like(m, s)$$

in FO, giving `likes` the type $\langle e, \langle e, t\rangle\rangle$ allows it to combine with the object first, as it should on a compositional analysis if `likes Sue` is a constituent of (3), and then with the subject, yielding

$$like(s)(m)$$

Similarly,

(4) Some students smoke.

could be rendered as

$$some(student)(smoke)$$

(where *some, student, smoke* are constants of the appropriate types), reflecting the fact that `some students` is a constituent of (4). So far there are no lambdas. But suppose *some* is not a constant but rather *defined* as $\lambda X \lambda Y \exists x(X(x) \wedge Y(x))$. Then (4) would be rendered

(5) $\lambda X \lambda Y \exists x(X(x) \wedge Y(x))(student)(smoke)$

which after two *lambda conversions* becomes

$$\exists x(student(x) \wedge smoke(x))$$

This is the standard FO translation of (4), but now obtained compositionally. In this chapter I focus on succinct formulation of truth conditions of quantified sentences (not so much on their compositional derivation), and on model-theoretic properties of quantifiers, and then the relational set-theoretic notation seems simplest.

There also a middle way: skip the lambdas but keep the functional rendering of quantifiers, using ordinary set-theoretic notation (as in e.g. Keenan and Stavi (1986)). This makes a type $\langle 1, 1\rangle$ quantifier a function mapping sets (N extensions) to type $\langle 1\rangle$ quantifiers (NP extensions), which is just how Dets work. In principle, one can choose the notation one prefers; it is usually straightforward to translate between them.[6]

[6] See, however, Keenan and Westerståhl (2011), pp. 876–7, for some additional advantages of the functional version.

0.3 Noun phrases

We already saw that quantified NPs, consisting of a determiner and a noun, are most naturally interpreted as type $\langle 1\rangle$ quantifiers, i.e. on each universe M, as sets of subsets of M. For example, the extension of `three cats` is the set of subsets of M whose intersection with the set of cats has exactly three elements, and the extension of `no students but Mary` is the set of subsets of M whose intersection with the set of students is the unit set {Mary}. What happens is just that the first argument (the restriction argument) of the type $\langle 1,1\rangle$ quantifier Q that the Det denotes is fixed to a given set A. The operation is called restriction or *freezing*: Q and A yield a type $\langle 1\rangle$ quantifier Q^A. Normally one has $A \subseteq M$, but in principle we should define the action of Q^A on any universe M. This is done as follows: for all M and all $B \subseteq M$,[7]

(6) $\quad (Q^A)_M(B) \iff Q_{A\cup M}(A,B)$

Next, there is a class of quantified NPs that do not freeze to a noun with fixed extension like `cat`, but instead to a variable noun like `thing`, which can be taken to denote the universe. Some of these are lexicalized as words in English. But the interpretation mechanism is just as in (6), except that $A = M$. For example,

(7) $\quad \mathit{everything}_M(B) \Leftrightarrow (\mathit{every}^{\mathit{thing}})_M(B) \Leftrightarrow (\mathit{every}^M)_M(B)$
$\Leftrightarrow \mathit{every}_M(M,B) \Leftrightarrow M \subseteq B \Leftrightarrow B = M$

Similarly, applying (6) we obtain

(8) $\quad \mathit{something}_M(B) \Leftrightarrow B \neq \emptyset$
$\mathit{nothing}_M(B) \Leftrightarrow B = \emptyset$
$\textit{at least three things}_M(B) \Leftrightarrow |B| \geq 3$
$\textit{most things}_M(B) \Leftrightarrow |B| > |M - B|$
etc.

Here we note that the first two are the standard universal and existential quantifiers of FO: $\forall$ and $\exists$. GQ theory started as generalizations of these: Mostowski (1957) considered type $\langle 1\rangle$ quantifiers that place conditions on the cardinalities of B and $M-B$, as all the quantifiers listed above do. But the connection to the semantics of NPs was not then a motivation.

[7] See Peters and Westerståhl (2006), ch. 4.5.5, for arguments why this is the correct definition, rather than, say, $(Q^A)_M(B) \Leftrightarrow A \subseteq M \;\&\; Q_M(A,B)$.

What about NPs that don't have Dets? A typical case are those denoting a single individual, in particular *proper names*. The most straightforward interpretation of a name is as an element of the universe: `Mary` denotes m. However, Montague devised a treatment of names as type $\langle 1 \rangle$ quantifiers too, partly for the reason that they are easily *conjoined* with quantified NPs, as in

(9) Mary and three students went to the party.

Thus, for each individual a, the type $\langle 1 \rangle$ quantifier I_a is defined as follows: for all M and all $B \subseteq M$,

(10) $(I_a)_M(B) \iff a \in B$

Here we have not required that $a \in M$; if not, $(I_a)_M$ is simply empty.[8] But if $a \in M$, $(I_a)_M$ is the principal filter (over M) generated by a.

The quantifiers I_a are called *Montagovian individuals*. (9) also illustrates that *boolean combinations* of quantfiers (of the same type) works smoothly and as expected: in the type $\langle 1 \rangle$ case:

(11) $(Q \wedge Q')_M(B) \iff Q_M(B)$ and $Q'_M(B)$
$(Q \vee Q')_M(B) \iff Q_M(B)$ or $Q'_M(B)$
$(\neg Q)_M(B) \iff$ not $Q_M(B)$

So in (9), `Mary and three students` is conjoined from the NPs `Mary` and `three students`, each interpretable as a type $\langle 1 \rangle$ quantifier, and the interpretation of the conjoined NP is the conjunction of these two. Similarly, we get the obvious interpretations of NPs like `John and Mary`, `Fred or Mary, but not Sue`, etc. Summing up, interpreting names as Montagovian individuals provides us with interpretations of conjoined NPs that would otherwise not be easily available: boolean operations are defined on quantifiers but not on elements of the universe.

This takes care of a vast majority of English NPs. A kind not mentioned so far are *bare plurals*, but these can be (roughly) treated as if they had a null (universal or existential) Det:

Firemen wear helmets ≈ All firemen wear helmets

Firemen are available ≈ Some firemen are available

[8] So on this analysis, if Mary is not in the universe of discourse, all positive claims about her (that she has a certain property) are false. Note that requiring that $a \in M$ changes nothing: for all a, all M, and all $B \subseteq M$, $a \in B \Leftrightarrow a \in M \,\&\, a \in B$.

But what about the following?

(12) Only firemen wear black helmets.

(13) Only John smokes.

If we allow that `John` denotes the singleton $\{j\}$, it might *seem* that these two sentences have the same form, namely,

only A are B,

and that `only` is a Det whose interpretation is $only_M(A, B) \Leftrightarrow \emptyset \neq B \subseteq A$. However, several things speak against idea. First, we can agree that (12) and (13) have the same form, but that the form is rather

only Q are B,

where Q is an NP. After all, `John` is an NP, and `firemen` too, if taken as a bare plural. And other NPs work as well:

(14) Only the three boys were rescued.

But then `only` modifies an NP and is not a Det, and the interpretation of `[only NP]` is not obtained along the lines of (6).[9] In the next section we will see another reason why `only` cannot be a Det.

Let me also mention that there are NPs formed by Dets applying to more than one N argument, as in

(15) More men than women smoke.

[9] What do these NPs mean, i.e. which type $\langle 1 \rangle$ quantifiers interpret them? Try:

(i) $(only\ Q^A)_M(B) \iff Q_M(A, B)\ \&\ B \subseteq A$

This actually gives correct truth conditions for (12)–(14), provided we (a) use the *decomposition* of I_a as $some^{\{a\}}$, i.e. we use the fact that $I_a = some^{\{a\}}$; (b) similarly decompose the existential reading of bare plurals. (If that reading is given by the quantifier $(C^{e,pl})_M(B) \Leftrightarrow C \cap B \neq \emptyset$, so $C = firemen$ in our example, then we have $C^{e,pl} = some^C$.) But this is complicated by the fact that such decomposition is not unique (see Westerståhl (2008) for more about decomposition), and also that `only` has other uses. For example,

(ii) Only ten boys were rescued.

has, besides the reading given by (i), the reading (with focus on `boys`) that exactly ten boys were rescued, but that others (e.g. girls) might also have been rescued. Then it is truth-conditionally equivalent to `Ten boys were rescued`, and `only` is rather a pragmatic addition that this number is remarkable in some way. Indeed, the complex semantics of `only` and `even` is crucially tied to focus phenomena, which are partly pragmatic; see Rooth (1996) for a survey.

This is quite naturally seen as a type $\langle 1,1,1\rangle$ quantifier *more_than* applied to two noun denotations, yielding—along lines generalizing (6)—a type $\langle 1\rangle$ quantifier interpreting the NP `more men than women`:

$$more\_than_M(A,B,C) \iff |A\cap C| > |B\cap C|$$

I will not deal further with such Dets here; see Keenan and Westerståhl (2011), sect. 19.2.3, for more examples and discussion.

0.4 Domain restriction

Sentences like

(16) Most students smoke.

have a perfectly clear constituent structure, i.e.

(16)′ $[_S\ [_{NP}\ [_{Det}$ most$]\ [_N$ students$]]\ [_{VP}$ smoke$]]$

or, schematically,

(16)″ $[QA]B$

So it is obvious that the *restriction* argument A of Q plays a very different syntactic role than the *nuclear scope* argument B. Is there a semantic counterpart to this?

Indeed there is, and this is the most characteristic trait of type $\langle 1,1\rangle$ quantifiers that interpret natural language Dets. Intuitively, the domain of quantification is restricted to A. Technically, this can be described via the model-theoretic notion of relativization. For any Q of type $\langle n_1,\ldots,n_k\rangle$, the *relativization* of Q is the quantifier Q^{rel} of type $\langle 1,n_1,\ldots,n_k\rangle$ defined by

$$(Q^{rel})_M(A,R_1,\ldots,R_k) \iff Q_A(R_1\cap A^{n_1},\ldots,R_k\cap A^{n_k})$$

In particular, for Q of type $\langle 1\rangle$, Q^{rel} has type $\langle 1,1\rangle$ and

(17) $(Q^{rel})_M(A,B) \iff Q_A(A\cap B)$

What Q^{rel} does is to take its first argument as a universe and describe the action of Q on that universe. That is, $Q^{rel}(A,\ldots)$ 'simulates' Q with A as its domain of quantification. This means that the idea that the first argument of a type $\langle 1,1\rangle$ Det interpretation Q provides the domain of quantification can be expressed as follows: *Q is the relativization of some type $\langle 1 \rangle$ quantifier.*

Historically, this crucial property of Det interpretations was approached in a different way. It was noticed early on (Barwise and Cooper, 1981; Higginbotham and May, 1981; Keenan, 1981) that these quantifiers have the property of *conservativity*: for all M and all $A, B \subseteq M$,

$$(\textsc{Conserv}) \quad Q_M(A, B) \iff Q_M(A, A \cap B)$$

This can be easily checked in each case, e.g. the following sentence pairs are not only equivalent but trivially so, in that the second sentence contains an obvious redundancy.

(18) a. Most students smoke.
b. Most students are students who smoke.

(19) a. All but five teams were disqualified.
b. All but five teams are teams that were disqualified.

(20) a. Q *As* are B
b. Q *As* are *As* that are B

Conserv rules out many type $\langle 1, 1 \rangle$ quantifiers that are quite natural from a logical or mathematical point of view, but cannot serve as interpretations of English determiners, for example,

(21) $more_M(A, B) \Leftrightarrow |A| > |B|$
$I_M(A, B) \Leftrightarrow |A| = |B|$ (the equicardinality or Härtig quantifier)

It also rules out `only` as a Det: interpreted as suggested in the previous section it would not be Conserv. Note that all these quantifiers are easily expressed in English; for example, $more_M(A, B)$ says `There are more As than Bs`, and $only_M(A, B)$ would say `There are Bs and all Bs are As`. But the point is that they do not interpret English *determiners.*

Conserv contains part of the idea of domain restriction, since it says in effect that the elements of $B - A$ do not matter for the truth value of $Q_M(A, B)$. But it says nothing about elements of the universe that are outside both A and B, i.e. in $M - (A \cup B)$. For example, it doesn't rule out a quantifier *unex* that behaves as *every* on universes with at most 100 elements, and as *some* on larger universes.

Although a quantifier Q may associate any local quantifier Q_M on a universe M, it seems reasonable to say that *unex*, even though it is

Conserv (and definable in first-order logic), is not the *same* on all universes, or that it is not *uniform*. The following property, introduced in van Benthem (1984) under the name of *extension*, goes a long way to capture the idea of sameness or uniformity over universes. It applies to quantifiers of all types; here is the type $\langle 1, 1\rangle$ case:

(Ext) If $A, B \subseteq M \subseteq M'$, then $Q_M(A, B) \Leftrightarrow Q_{M'}(A, B)$.

In other words, what the universe is like outside the arguments A and B doesn't matter. This rules out quantifiers like *unex*. But all the other type $\langle 1, 1\rangle$ quantifiers shown so far are Ext. And it is easy to see that relativized quantifiers are always Ext. Among the type $\langle 1\rangle$ quantifiers looked at so far, all are Ext *except* $\forall$ (*everything*) and *most things*. Significantly, the latter two involve a noun *thing* that refers to the universe, and in these cases (as opposed to *something* or *nothing*), whether $Q_M(B)$ holds or not depends also on the *complement* $M - B$. For example, $\forall_M(B)$ says that $M - B$ is empty, and so Ext fails.

For Det interpretations, Ext should be part of the idea of domain restriction, even if the rationale for Ext goes far beyond that. If the truth value of $Q_M(A, B)$ could change when the universe M is extended, we could hardly say that A was the domain of quantification. Now it turns out that Conserv and Ext together exactly capture domain restriction. The following fact is essentially trivial but basic, so I will give the proof.

Fact 0.1 *A type* $\langle 1, 1\rangle$ *quantifier is* Conserv *and* Ext *if and only if it is the relativization of a type* $\langle 1\rangle$ *quantifier.*

Proof If Q is of type $\langle 1\rangle$ it is straightforward to check that Q^{rel} is Conserv and Ext. Conversely, suppose the type $\langle 1, 1\rangle$ quantifier Q' is Conserv and Ext. Define Q of type $\langle 1\rangle$ by

$$Q_M(B) \Leftrightarrow Q'_M(M, B)$$

Then we have, for all M and all $A, B \subseteq M$,

$$\begin{aligned}
(Q^{rel})_M(A, B) &\Leftrightarrow Q_A(A \cap B) && \text{(def. of } Q^{rel}\text{)} \\
&\Leftrightarrow Q'_A(A, A \cap B) && \text{(def. of } Q\text{)} \\
&\Leftrightarrow Q'_M(A, A \cap B) && \text{(Ext)} \\
&\Leftrightarrow Q'_M(A, B) && \text{(Conserv)}
\end{aligned}$$

That is, $Q' = Q^{rel}$. □

So we have a clear model-theoretic characterization of the special semantic role of the restriction argument of Dets. A model $\mathcal{M} = (M, A, B)$ can be depicted as follows.

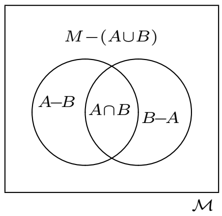

Figure 0.1 A model of type $\langle 1, 1\rangle$

CONSERV says that $B-A$ doesn't matter for whether $Q_M(A, B)$ holds or not. EXT says that $M-(A\cup B)$ doesn't matter. The only sets that matter are $A-B$ and $A\cap B$, both subsets of A. That's what it means that Q restricts the domain of quantification to its first argument.

In addition, it seems that EXT is a kind of semantic universal: All 'reasonable' quantifiers, *except* some of those who involve a predicate like *thing*, satisfy it. EXT also lets us simplify notation and write $Q(A, B)$ instead of $Q_M(A, B)$; a practice I will follow whenever feasible.

0.5 Quantity

You would think quantifiers had something to do with quantities, and indeed we see directly that most of the Det interpretations in (2) are perfectly good answers to the question

(22) How many As are B?

So it is the *number of elements* in the concerned sets that matter, not the elements themselves. Formally, if we have two models as in Fig. 0.2 the requirement is

(ISOM) If the corresponding four sets in Figure 0.2 have the same cardinality, then $Q_M(A, B) \Leftrightarrow Q_{M'}(A', B')$.

This property is called *isomorphism closure* in model theory, since the antecedent amounts the existence of an isomorphism from $\mathcal{M}$ to $\mathcal{M}'$.

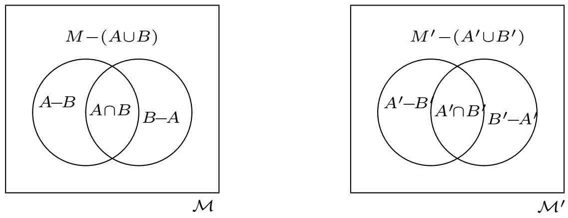

Figure 0.2 Two models of type $\langle 1, 1\rangle$

The notion of isomorphism applies to models of any type, and hence so does Isom.

In the case of a type $\langle 1, 1\rangle$ quantifier satisfying Conserv and Ext, Isom amounts to the following:

(23) If $|A–B| = |A'–B'|$ and $|A\cap B = |A'\cap B'|$ then $Q(A,B) \Leftrightarrow Q(A',B')$.

This is why $Q(A, B)$, for Conserv, Ext, and Isom Q, answers question (22). It means that these quantifiers can be seen as *binary relations between* numbers; in the case of finite models between natural numbers. This is a huge simplification; recall that by definition quantifiers are operators that with each universe (non-empty set) associate a second-order relation on that set. Now this is reduced to a (first-order) relation between numbers, with no mention of universes. For example, with $|A–B| = m$ and $|A \cap B| = n$,

all $(m, n) \Leftrightarrow m = 0$
some $(m, n) \Leftrightarrow n > 0$
most $(m, n) \Leftrightarrow n > m$
an odd number of $(m, n) \Leftrightarrow n$ is odd

This also simplifies the model-theoretic study of the expressive power of quantifiers. It follows from Fact 0.1 that, under Isom, Q and Q^{rel} define the same binary relation between numbers (where Q has type $\langle 1\rangle$), and it is much easier to obtain results for type $\langle 1\rangle$ quantifiers than for any other types.

A weaker version of Isom, called Perm, has the same definition except that $M' = M$. This is closure under *automorphisms* or, equivalently, under *permutations* (since every permutation of the universe M induces an automorphism on $\mathcal{M}$). If one is working with a fixed universe of discourse, Perm is the natural choice. One easily construes artificial examples of quantifiers that are PERM but not ISOM. (For example, let a be a fixed object, and let $Q_M = some_M$ if $a \in M$ and $Q_M = every_M$ otherwise.) In the presence of EXT, however, the difference disappears; one can show that the following holds for Q of any type.

Fact 0.2 *If Q is* EXT *and* PERM, *then Q is* ISOM.

But some of the Det interpretations in (2) are not ISOM (not even PERM): `Henry's, some students', every_except Mary`, and likewise the Montagovian individuals I_a. This makes perfect sense: all of these depend on a fixed property like being a student, or a fixed individual like Mary. Permuting the elements of the universe may map *student* to *cat*, or Mary to Henry. Also, these quantifiers occur less naturally as answers to (22); cf.

(24) a. How many dogs are in the pen?
b. At least three/no/more than half of the dogs are in the pen.
c. ? Henry's dogs are in the pen.

From a logical point of view, one might then prefer to generalize these second-order relations by taking the additional set or individual as extra arguments. Then, *some students'* would be of type $\langle 1, 1, 1\rangle$, and *every_except Mary* of type $\langle 1, 1, 0\rangle$, where 0 now stands for an indivdual. These quantifiers would all be ISOM. Logicians usually look at model-theoretic objects 'up to isomorphism', and indeed closure under isomorphism was part of Mostowski's and Lindström's original definition of a generalized quantifier. But from a linguistic perspective, the type should correspond to the syntactic category of the relevant expression. As long as there are independent reasons to think of `Henry's, some students', every_except Mary` etc. as Dets, one would want to to interpret them as type $\langle 1, 1\rangle$ quantifiers.

0.6 Negation

The most common occurrence of negation in English is not the logician's sentential negation, 'it is not the case that', but VP negation. (25b) can be true when (25a) is false.

(25) Two-thirds of the students smoke.
a. Two-thirds of the students don't smoke.
b. It is not the case that two-thirds of the students smoke.

VP negation corresponds to a natural boolean operation on type $\langle 1 \rangle$ quantifiers: In addition to (11), we have

(26) $\quad (Q\neg)_M(B) \iff Q_M(M - B)$

This operation is called *inner negation* or, in Keenan's terminology, *post-complement*. Like the other boolean operations, it applies also to Conserv and Ext type $\langle 1, 1 \rangle$ quantifiers, and combined with normal (*outer*) negation it yields a natural notion of a *dual* quantifier; in the type $\langle 1, 1 \rangle$ case:

(27) $\quad (Q\neg)(A, B) \iff Q(A, A - B)$

(28) $\quad Q^d = \neg(Q\neg) = (\neg Q)\neg$

For example, *no* is the inner negation of *all*, *some* is the dual, and *not all* is the outer negation. Medieval Aristotle scholars noticed early on that these four quantifiers can be geometrically displayed in a *square of opposition*. In fact, we can now see that every Conserv and Ext type $\langle 1, 1 \rangle$ quantifier Q spans a square of opposition $\textit{square}(Q) = \{Q, Q\neg, Q^d, \neg Q\}$:[10]

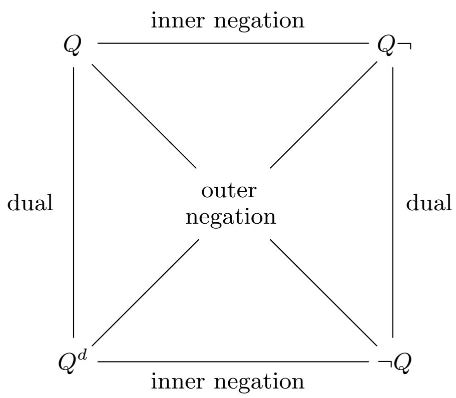

Figure 0.3 *square*(Q)

Outer and inner negation and dual are all *idempotent*: $Q = \neg\neg Q = Q\neg\neg = Q^{dd}$. They interact with $\wedge$ and $\vee$ as follows:

[10] This is not quite an aristotelian-type square, which instead of inner negation along the top side has the relation of *contrariety* (Q and Q' are contraries if $Q(A, B)$ and $Q'(A, B)$ can never both be true), and in fact differs along all the sides of the square; only the diagonals are the same. For a comparison between the two kinds of square, and the properties of several examples of (modern) squares of generalized quantifiers, see Westerståhl (2012).

(29) a. $\neg(Q \wedge Q') = \neg Q \vee \neg Q'$ and $\neg(Q \vee Q') = \neg Q \wedge \neg Q'$ (de Morgan laws)
b. $(Q \wedge Q')\neg = Q\neg \wedge Q'\neg$ and $(Q \vee Q')\neg = Q\neg \vee Q'\neg$
c. $(Q \wedge Q')^d = Q^d \vee Q'^{\,d}$ and $(Q \vee Q')^d = Q^d \wedge Q'^{\,d}$

Using these facts, it is easy to verify that every quantifier in *square*(Q) spans the same square (so two squares are either disjoint or identical), which always has either 4 or 2 members.

Inner negations and duals of English Dets are often expressible as other Dets, without using explicit boolean connectives. For example, one checks that $(all)^d$ = *some*, (*at most three*)$\neg$ = *all but at most three*, (*the ten*)$\neg$ = *none of the ten*, (*at least two-thirds of the*)^d = *more than one-third of the*, (*all _ except Mary*)$\neg$ = *no _ except Mary*, (*exactly half the*)$\neg$ = *exactly half the*. In short, the square of opposition is a very useful tool for understanding the relations between the various forms of negation occurring in natural languages.

As the last example above shows, we can have $Q = Q\neg$ for naturally occurring quantifiers (and so *square*(Q) has 2 members). Keenan has observed that there are also many less obvious examples; the equivalence of the following pair requires a small calculation:

(30) a. Between 10 and 90 percent of the students left.
b. Between 10 and 90 percent of the students didn't leave.

That is, (*between 10 and 90 percent of the*)$\neg$ = *between 10 and 90 percent of the*.

Can the vertical sides of the square also be collapsed, i.e. can we have $Q = Q^d$, or equivalently, $\neg Q = Q\neg$? Such quantifiers are called *self-dual* in Barwise and Cooper (1981). The answer to this question reveals an interesting difference between global and local quantifiers. For, in contrast with the collapse $Q = Q\neg$, which as we saw does occur for certain common global quantifiers, *there are no interesting self-dual global quantifiers*. The next fact provides evidence for this (somewhat vague) claim. I formulate it here for a type $\langle 1 \rangle$ quantifier Q.

Fact 0.3 *If Q is either (i)* Isom*, or (ii) a Montagovian individual I_a, or (iii) of the form $(Q_1)^A$ for some* Conserv *type $\langle 1,1 \rangle$ Q_1, then Q is not self-dual.*[11]

[11] *Outline of proof*: For (i), choose M and $B \subseteq M$ s.t. $|B| = |M - B|$. Then $Q_M(B) \Leftrightarrow Q_M(M - B|)$ by Isom, which contradicts $Q_M = (Q^d)_M$. (As Keenan (2005) observes, this argument works for any local Perm quantifier Q_M, as long as $|M|$ is either even or infinite.) For (ii), choose M s.t. $a \notin M$. Then $(I_a)_M$ is

Thus, most (all?) common NP interpretations are not globally self-dual. This may seem surprising, since precisely in case (ii), self-duality has been cited as a significant property of Montagovian individuals! The explanation is that a *local* quantifier $(I_a)_M$ is self-dual if $a \in M$, for then we have: $(\neg I_a)_M(B) \Leftrightarrow a \notin B \Leftrightarrow a \in M - B \Leftrightarrow (I_a\neg)_M(B)$. As Barwise and Cooper note, this corresponds to the fact that *names lack scope wrt negation*: in contrast with (25), the following sentences are equivalent:

(31) a. Ruth doesn't smoke.
b. It is not the case that Ruth smokes.

In fact, the interpretation of names as Montagovian individuals is also easily seen to explain why they lack scope wrt all boolean operations; cf. the equivalence of

(32) a. Bill or John smokes.
b. Bill smokes or John smokes.

Frans Zwarts has shown that lacking scope wrt boolean operators holds for *exactly* the Montagovian individuals.[12]

So here we have another case of a linguistically significant phenomenon with a clear model-theoretic counterpart. But this time the property is local, not global.

0.7 Polarity and monotonicity

Natural languages have expressions—called *negative polarity items* (NPIs)—that seem to occur only (with a particular sense) in negative contexts, and as it were make the negative claim as strong as possible. Two prime English examples are `ever` and `yet`:

(33) a. Susan hasn't ever been to NYC.
b. *Susan has ever been to NYC.

(34) a. Henry hasn't read the morning paper yet.
b. *Henry has read the morning paper yet.

the trivially false quantifier on M, but $(I_a)^d_M$ is the trivially true quantifier on M. For (iii), choose M s.t. $A \cap M = \emptyset$. Then an easy calculation, using the conservativity of Q_1, shows that $Q_M = (Q\neg)_M$, contradicting self-duality.

12 van Benthem (1989) and Zimmermann (1993) have further discussion and results about scopelessness.

Other examples include `any`, and various idioms such as `give a damn` and `budge an inch`. An obvious linguistic concern is to identify (a) the class of NPIs, and (b) the positions in which they occur. It is with (b) that model-theoretic semantics, and in particular GQ theory, turns out to be useful.[13]

It is well-known that NPIs also occur in certain positions not in the scope of an explicit negation:

(35) a. Less than half of my friends have ever been to NYC.
b. *At least half of my friends have ever been to NYC.

(36) a. No one here has read the morning paper yet.
b. *Someone here has read the morning paper yet.

So what is 'negative' about `less than half`? The proper generalization appears to turn on the concept of *monotonicity*.[14]

A function f from an ordered set $(X, \leq_1)$ to an ordered set $(Y, \leq_2)$ is

increasing if $x \leq_1 y$ implies $f(x) \leq_2 f(y)$;
decreasing if $x \leq_1 y$ implies $f(y) \leq_2 f(x)$.

Sometimes *monotone* is used synonymously with '(monotone) increasing'; here it will mean 'either increasing or decreasing'. Now negation is a prime example of a decreasing function; then $X = Y$ is a class of propositions and $\leq_1 = \leq_2$ is implication. (Alternatively, $X = Y$ is the set of truth values $\{0, 1\}$ and $\leq_1 = \leq_2$ is the usual non-strict order among them.) And most quantifiers interpreting NPs or Dets are monotone in some respect. More precisely, an Ext type $\langle 1, 1\rangle$ quantifier Q is

right increasing if for all A, B, B', $Q(A, B)$ & $B \subseteq B'$ implies $Q(A, B')$,

and similarly for *right decreasing*, *left increasing*, *left decreasing*, and correspondingly for type $\langle 1\rangle$ quantifiers (without the 'right' and 'left').[15]

[13] There are also *positive* polarity items, with a behavior partly symmetric to NPIs, such as `already`:

(i) a. Bill has already heard the news.
b. *Bill hasn't already heard the news.

I can only scratch the surface of the complex issues surrounding NPIs and PPIs here; see Ladusaw (1996) and Peters and Westerståhl (2006), ch. 5.9, for surveys and relevant references.

[14] NPIs also occur in questions, comparatives, antecedents of conditionals, complements of phrases like `it is surprising that`. Here I focus on their occurrence in quantified contexts.

[15] So here $\leq_1$ is $\subseteq$, but $\leq_2$ is still the implication order.

For example, *every* is left decreasing and right increasing, *at least five* is left and right increasing, *most* is right increasing but not left monotone, and the same holds for *the ten*. Numerical quantifiers like *exactly five*, *between two and seven*, *either at least three or no* are not themselves monotone but boolean combinations of monotone (in fact right and left increasing) quantifiers. To find thoroughly non-monotone Det denotations we need to look at examples like *an even number of*.

The monotonicity behavior of an NP of the form [Det N] is determined by that of the Det, not by that of the N. This fact has a straightforward semantic explanation (cf. section 0.4): a type $\langle 1 \rangle$ quantifier Q is increasing (decreasing) iff Q^{rel} is right increasing (decreasing). Also note that the behavior of Q determines exactly the behavior of the other quantifiers in $square(Q)$. For example, if Q (type $\langle 1, 1 \rangle$) is right decreasing and left increasing, then Q^d is right and left increasing; the reader can easily figure out the other correspondences that hold.

Now, what is characteristic of NPIs seems to be that they occur in *decreasing* contexts, and as we saw in (35a) this can happen without there being any explicit negation: `less than half of my friends` is decreasing but not `at least half of my friends`. So to the extent this is correct we again have a model-theoretic property that explains or at least systematizes a linguistic phenomenon.

In fact, much more can (and should) be said about polarity. For just one example, there is a difference between `ever` and `yet`: the former is fine in all decreasing contexts, but not the latter:

(37) a. None of my friends have seen *Alien* yet.
b. *At most three of my friends have seen *Alien* yet.

Zwarts argued that `yet` requires the stronger property of anti-additivity: An Ext type $\langle 1 \rangle$ quantifier Q is

anti-additive if for all B, C, $Q(B)$ & $Q(C)$ iff $Q(B \cup C)$.

This is equivalent to Q^{rel} being *right anti-additive*: $Q^{rel}(A, B)$ & $Q^{rel}(A, C)$ iff $Q^{rel}(A, B \cup C)$. It is clear that being (right) anti-additive implies being (right) decreasing, but the converse fails. In fact, Peters and Westerståhl (2006), ch. 5.9.4, show that over finite universes, very few Isom quantifiers Q^{rel} are right anti-additive: essentially only *no*, *none of the k (or more)*, and disjunctions of these. Model-theoretic results like this make it feasible to empirically test various hypotheses about the distribution of NPIs, but I shall leave the topic of NPIs here.

Monotonicity is relevant to natural language semantics far beyond establishing the distribution of NPIs. Most conspicuously, it pervades much of everyday *reasoning*. Aristotle's syllogisms express various kinds of monotonicity. For example,

$$\begin{array}{lll} \textit{all } BC & \qquad & \textit{no } BC \\ \underline{\textit{all } AB} & & \underline{\textit{all } AB} \\ \textit{all } AC & & \textit{no } AC \end{array}$$

say that *all* is right increasing and *no* is left decreasing, respectively. If the syllogistics is seen as a logic calculus for monadic predicates, it is poor in expressive power. But, as van Benthem (2008) points out, if we instead see it as recipes for *one-step monotonicity reasoning* (allowing A, B, C to be complex predicates), it takes on a new flavor. This is a leading idea in the program of *natural logic*: making inferences directly on natural language forms, without first translating them into some formal language.[16] For monotonicity reasoning, one can systematically, during a syntactic analysis, mark certain predicate occurrences as increasing (+) or decreasing (–), allowing the corresponding inferences at those positions. For example, we would write (leaving out the simple structural analysis)

(38) a. All students$^-$ jog$^+$.
 b. Most professors smoke$^+$.

Here (38a) indicates that we may infer e.g. `All graduate students jog`, or `All students jog or swim`. There is no marking on `professors` in (38b), since *most* is not left monotone, but since it is right increasing, there is a + on `smoke`.

Direct monotonicity inferences are simple, but combination with other natural modes of reasoning can give quite intricate results. The following example is adapted from Pratt-Hartmann and Moss (2009):

(39) a. All skunks are mammals
 b. *Hence*: All people who fear all who respect all skunks fear all biologists who respect all mammals

To see that the conclusion really follows, start with the logical truth:

(40) All biologists$^-$ who respect$^-$ all skunks$^+$ respect$^+$ all skunks$^-$

[16] For example, Sommers (1982) and Sanchez Valencia (1991). van Benthem (2008) gives a good overview.

Note that relation occurrences too can have signs. For example, changing the first occurrence of `respect` to `respect and admire` preserves validity. But this is not used in (39). Instead we use the + on the first occurrence of `skunks` to obtain

(41) a. All skunks are mammals
b. *Hence*: All biologists who respect all mammals respect all skunks

(41) is valid, and validity is preserved under appropriate replacement of predicates. So replace `skunks` by `biologists who respect all mammals`, replace `mammals` by `(individuals who) respect all skunks`, replace `biologists` by `people`, and replace `respect` by `fear`. The result is

(42) a. All biologists who respect all mammals respect all skunks
b. *Hence*: All people who fear all who respect all skunks fear all biologists who respect all mammals

Thus, using, besides monotonicity, *substitution* and *transitivity* of consequence, we obtain (39).

But isn't this *logic* rather than *semantics*? Actually the dividing line is not so clear, and many semanticists take the meaning of an expression to be essentially tied to the valid inferences containing it. The point here is just that these inferences can often be 'read off' more or less directly from surface structure, without translation into a formal language like first-order logic or intensional type theory.[17]

Finally, consider

(43) a. More than 90 percent of the students passed the exam
b. At least 10 percent of the students play tennis
c. *Hence*: Some student who plays tennis passed the exam

This doesn't quite look like monotonicity reasoning of the previous kind. Why does the conclusion follow from the premisses? The first pertinent observation is that *at least 10 percent of the* is the dual of *more than 90 percent of the*. So the pattern is

[17] But one can also make a logical study of syllogistic languages, extended with names, transitive verbs, adjectives, etc. In a series of papers Larry Moss has pursued this (see e.g. Pratt-Hartmann and Moss (2009); Moss (2010)), with particular attention to formats for (complete) axiomatization, and how far these fragments of first-order logic can remain *decidable*, unlike *FO* itself. (39) is a variant of an example mentioned in Pratt-Hartmann and Moss (2009); this form of reasoning is treated in detail in Moss (2010).

(44) a. $Q(A, B)$
b. $Q^d(A, C)$
c. *Hence*: $some(A \cap C, B)$

Now, Barwise and Cooper (1981), Appendix C, noted that if Q is right increasing, this pattern is always valid. And *more than 90 percent of the* is indeed right increasing. Further, Peters and Westerståhl (2006), ch. 5.8, showed that the pattern actually *characterizes* that property: A CONSERV quantifier Q is right increasing iff it validates (44). So (43) too turns out to essentially involve monotonicity, but in addition also inner and outer negation.

For determiners, right monotonicity is much more common than the left variant. Indeed, Westerståhl (1989) showed that over finite universes, each left monotone CONSERV, EXT, and ISOM quantifier is first-order definable, which entails that there are countably many such quantifiers, whereas there are uncountably many right monotone quantifiers with the same properties. And van Benthem (1984) showed that under these conditions plus a natural non-triviality requirement, the only *doubly* (both left and right) monotone quantifiers are the four ones in $square(all)$. Moreover, Barwise and Cooper (1981) proposed as one of their *monotonicity universals* that every Det denotation which is left monotone is also right monotone, a generalization which seems to be borne out by the facts.

There is another monotonicity observation worth mentioning. Almost all right increasing Det denotations in fact have a stronger property, called *smoothness*.[18] It is actually a combination of two weaker *left* monotonicity properties, one increasing and one decreasing. Recall that Q is left decreasing when $Q(A, B)$ is preserved if A is decreased. The weaker property is: $Q(A, B)$ is preserved if A is decreased *outside* B. That is, if $A' \subseteq A$ but $A \cap B = A' \cap B$, then $Q(A', B)$. For lack of a better name, I will say that Q is *left outside decreasing*. Likewise, while Q being left increasing means that $Q(A, B)$ is preserved if A is increased, the weaker property of being *left inside increasing* is that $Q(A, B)$ is preserved if A is increased *inside* B, i.e. so that $A - B = A' - B$. Q is *smooth* iff it is left outside decreasing *and* left inside increasing. It is fairly straightforward to show that if a CONSERV Q is smooth, it is right increasing, but the converse is far from true. However, most common right increasing Det denotations are in fact smooth; for example,

[18] In Peters and Westerståhl (2006). It was first identified under the name of *continuity* in van Benthem (1996).

right increasing numerical quantifiers (boolean combinations of *at least* n), right increasing proportional quantifiers (such as *at least* m/n*ths of the*), and right increasing possessive quantifiers (see section 0.10).[19]

The following inference illustrates the property of being left outside decreasing (given that only men can join the men's soccer team), without being left monotone:

(45) a. At least one-third of the students joined the men's soccer team
b. *Hence*: At least one-third of the male students joined the men's soccer team

From a logical point of view, the two right monotonicity properties, plus the two properties constituting smoothness, plus two obvious variants of these—*left outside increasing*, and *left inside decreasing*, respectively—are the basic six monotonicity properties: all others, like smoothness and left monotonicity, are combinations of these. Furthermore, properties that one would have thought have nothing to do with monotonicity result from combining them. I end this section with the following slightly surprising fact (the proof is not difficult; see Peters and Westerståhl (2006), ch. 5.5):

Fact 0.4 *A* CONSERV *quantifier Q is symmetric (i.e. $Q_M(A, B)$ implies $Q_M(B, A)$ for all M and all $A, B \subseteq M$) if and only if it is both left outside decreasing and left outside increasing.*

Already Aristotle noted that *some* and *no* are symmetric ('convertible'), in contrast with *all* and *not all*. But inferences illustrating this, such as

(46) a. <u>No fish are mammals</u>
b. No mammals are fish

were not part of the syllogistic, which as we saw dealt with right and left monotonicity.

[19] Most but not all: quantifiers requiring that $|A|$ is not decreased, like those of the form *at least k of the n (or more)*, are exceptions.

0.8 Symmetry and existential there sentences

Many Dets besides `some` and `no` denote symmetric (SYMM) quantifiers, for example, `at least five, no more than seven, between three and nine, finitely many, an odd number of, no...but Mary`. And many others are *co-symmetric*, meaning that their inner negation is symmetric; `every, all but five, all...except John`. (Co-)symmetry is preserved under conjunction and disjunction, and if Q is symmetric, so is $\neg Q$, whereas $Q\neg$ and Q^d are co-symmetric. Quantifiers that are neither symmetric nor co-symmetric are, for example, proportionals like *most*, *fewer than one-third of the*, definites like *the ten*, and possessives like *Mary's* and *some student's*.

Another formulation of symmetry is the following (Keenan): Q is *intersective* if the truth value of $Q_M(A,B)$ depends only on the set $A \cap B$, that is (assuming EXT in what follows, for simplicity),

(INT) If $A \cap B = A' \cap B'$, then $Q(A,B) \Leftrightarrow Q(A',B')$.

We have:

Fact 0.5 *If Q is* CONSERV*, the following are equivalent:*

a. *Q is symmetric.*
b. *Q is intersective.*
c. $Q(A,B) \Leftrightarrow Q(A \cap B, A \cap B)$ [20]

In fact, all the above examples of symmetric/intersective quantifiers, except *no... but Mary*, have the stronger property of being what Keenan calls *cardinal*: $Q(A,B)$ depends only on the number of elements in $A \cap B$, i.e. only on $|A \cap B|$.

One reason to be interested in symmetry is that it seems to be tied to the analysis of so-called *existential there* sentences. Compare the following.

(47) a. There is a cat here.
b. *There are most cats here
c. There are no signatures on these documents.
d. *There are the signatures on these documents.
e. There are over a hundred religions.

[20] *Proof*: $a \Rightarrow c$: Suppressing the universe M for simplicity, we have: $Q(A,B) \Leftrightarrow Q(A, A \cap B)$ (CONSERV) $\Leftrightarrow Q(A \cap B, A)$ (SYMM) $\Leftrightarrow$ $Q(A \cap B, A \cap B)$ (CONSERV). $c \Rightarrow b$: Immediate. $b \Rightarrow a$: Since $A \cap B = B \cap A$ it follows by INT (with $A' = B$ and $B' = A$) that $Q(A,B) \Rightarrow Q(B,A)$.

f. *There are more than 75 percent of the religions.
g. There are ten people coming.
h. *There are the ten people coming.

The form is

(48) `there be` [pivot NP] ([coda]),

or, in more semantic terms,

(49) `there be` $Q\ A\ (B)$

In (47), the optional *coda* is present in (47a), (47b) and (47g), (47h). But not all the sentences are well-formed. A classical issue in linguistics is to characterize the ones that are. An early idea is that *definite* NPs (or Dets) are ruled out. The notion of definiteness is another can of worms (see section 0.9), but one can see that it cannot provide the whole explanation, since `most cats` and `more than 75 percent of the religions` are in no sense definite. Milsark (1977) concluded that the acceptable pivot NPs were not quantifiers but "cardinality words", and that putting genuinely quantified NPs there would result in a "double quantification", since `there be` already expresses quantification, which would be uninterpretable.

This was an attempt at a *semantic* explanation of existential there acceptability, though not in terms of precise truth conditions. Keenan (1987) gave a precise compositional account, inspired by some of Milsark's insights, but rejecting both the idea that acceptable pivot NPs are not quantified and that `there be` expresses quantification. Instead, acceptability turns, according to Keenan, on a model-theoretic *property* of quantifiers.

In fact, Keenan's analysis was proposed as an alternative to another celebrated semantic (and pragmatic) account of existential acceptability: the one in Barwise and Cooper (1981) in terms of so-called *weak* and *strong* quantifiers. (These terms were used by Milsark, but Barwise and Cooper redefined them.) A detailed comparison of the two approaches, in theoretical as well as empirical respects, is given in Peters and Westerståhl (2006), ch. 6.3. Here I will only outline the main points in Keenan's proposal.

The natural meaning of a sentence of the form (48) or (49) *without* a coda, like (47c) and (47e), is

$$Q_M(A, M)$$

which can be read 'Q^A exist'. So existence is not quantification but the predicate *exist*, whose extension is the universe M. When a coda is present, as in (47a) and (47g), it is equally clear that the reading is

(50) $Q_M(A \cap B, M)$

This is the intended *existential reading* of (acceptable) existential there sentences. Now, *prima facie* it is not trivial to obtain this reading by a compositional analysis of (48). Keenan's idea is that the compositional analysis always yields

(51) $(Q^A)_M(B)$, i.e. $Q_M(A, B)$

Then he stipulates that the acceptable quantifiers are exactly those for which these two are equivalent:

(52) $Q_M(A, B) \Leftrightarrow Q_M(A \cap B, M)$

But now it readily follows from Fact 0.5 that under Conserv, Q satisfies (52) if and only if Q is symmetric (intersective). So it is *symmetry*, according to this proposal, that characterizes the Dets of the quantified pivot NPs that can appear in existential there sentences. And this is not just a generalization from empirical data. It comes out of a natural compositional analysis, with the result that exactly for symmetric quantifiers do we obtain the intended existential reading.

Note that Milsark's "cardinality words" denote cardinal quantifiers, which are all symmetric. But also a non-Isom symmetric quantifier like *no... except Mary* is acceptable:

(53) There is no graduate student except Mary present.

On the other hand, proportional quantifiers like *most*, co-symmetric quantifiers like *every* and *all but seven*, and definites like *the ten*, are all ruled out, as they should be. In general, it is rather striking how well this simple model-theoretic criterion fits our intuitions about the meaning of existential there sentences.[21]

[21] Which is not to say that the analysis is unproblematic. Peters and Westerståhl (2006) discuss the following problems with Keenan's proposal: (i) It gives wrong predictions for proportional readings of `many` and `few`, which seem to be fine in existential there sentences; (ii) It assigns meanings to existential there sentences with non-symmetric Dets that these sentences do not have; (iii) If not emended, it gives wrong results for complex pivot NPs like `at least two of the five supervisors`, `several people's ideas`, which, although not symmetric, actually do occur (with the existential reading) in existential there sentences.

0.9 Definites

I have already used the common label *definite* for Dets like `the ten` (and correspondingly NPs like `the ten boys`). So what is definiteness? This is something linguists still debate, and overviews such as Abbott (2004) will present proposals, counter-examples, new proposals, etc. but no final definition. As Barbara Abbott concludes her survey paper: "As so frequently seems to be the case, grammar is willfully resistant to attempts at tidy categorization." My aim in this brief section is not to offer a new proposal, only to see to what extent GQ theory may help.

The above quote indicates that definiteness is a morphosyntactic category. Even so there might be a semantic and even model-theoretic *correlate*. But if there is no definition of morphosyntactic definiteness, it is hard to even start looking for such a correlate. There are clear cases—`John, the ten boys` are definite, `a tall man` is indefinite—but no definition. The most common *criterion* used is, in fact, unacceptability in existential there sentences. Could we use that as a definition? Then, assuming Keenan's analysis of these sentences to be largely correct, we *would* have a nice model-theoretic counterpart: non-symmetry.

But presumably no semanticist would accept that as a definition. For example, it would make the Det `most` definite, which few think it is. The criterion only works in some cases. Moreover, it is hardly syntactic. The sentence

(54) There is the problem of cockroaches escaping.

is fine as a follow-up to "Housing cockroaches in captivity poses two main problems," only *not in its existential reading.* So to get the right result—that `the problem of cockroaches escaping` is definite—we need to appeal to meaning after all.

Even if there is a fairly robust concept of purely *morphosyntactic definiteness*—one syntacticians recognize when they see it—there are also notions of definiteness that rely on the *meanings* of Dets and NPs. Most commonly, these are expressed in terms of *familiarity* and *uniqueness.* Familiarity is the idea that definite NPs refer back to something already existing in the discourse, whereas indefinites introduce new referents. Uniqueness is that there is a unique thing (in some sense) referred to by definite NPs.

Familiarity is a problematic criterion. To identify the things that are *available*, or *salient*, in the discourse universe is notoriously a pragmatic affair. Consider

(55) A woman came into the room and sat down. A cat jumped up in her lap/?on her books/??on her car.

We can easily see which variants are more easily interpretable, even if only the woman herself was explicitly introduced in the discourse, but what are the precise conditions?

Uniqueness seems more promising—if we drop the restriction to singular NPs and allow reference to sets or groups as well. Suppose you ask whether `most cats` is definite or not. We cannot even begin to apply the familiarity criterion, since there is no reasonable sense in which this NP introduces anything. But precisely for that reason, it fails to satisfy uniqueness.

Even if we grant a notion of, say, *pragmatic definiteness*, it seems clear that we also want a *semantic* notion. After all, none of the three variants of (55) is ungrammatical or meaningless. There is no semantic difference between `her lap`, `her books`, and `her car`, with respect to definiteness; all differences come from the surrounding discourse. So what is it about these three NPs that makes them (semantically) definite, in contrast with `most cats`? The answer seems simple: They are *referring expressions.*

Recall that we interpret NPs, on a given universe, as sets of subsets of that universe. In what way can we get *reference* out of such a set of subsets? If we include plural reference, i.e. reference to a collection or set of individuals, then GQ theory has a precise proposal, first made in Barwise and Cooper (1981):

> A type $\langle 1,1\rangle$ quantifier Q is (semantically) *definite* iff for each M and each $A \subseteq M$, either $(Q^A)_M$ is empty,[22] or there is a non-empty set $X \subseteq M$ such that $(Q^A)_M$ is the filter generated by X, i.e. $(Q^A)_M = \{B \subseteq M\colon X \subseteq B\}$.

Accordingly, a Det is (semantically) definite if it denotes a definite quantifier, in which case also NPs of the form [Det N], and their denotations, are definite. And we can extend the definition in an obvious way to non-quantified NP denotations, allowing us to call e.g. Montagovian individuals definite.

[22] Barwise and Cooper instead treat $(Q^A)_M$ as *undefined* in this case, in order to capture the idea that the existence of a generator is a *presupposition.* This plays a role in their account of existential there sentences, but not for what I will have to say here about definiteness. See Peters and Westerståhl (2006), chs. 4.6 and 6.3, for discussion.

Note first that the definition has a local character: in principle it would allow different generator sets X on different universes. However, Peters and Westerståhl (2006) prove (Proposition 4.10) that if Q is Conserv, Ext, and definite, then $(Q^A)_M$ is generated by the *same* set whenever it is non-empty. And this is what allows us to regard Q^A as referring: if it refers at all, it refers to that generator. Put differently, when $(Q^A)_M$ refers, it refers to $\cap(Q^A)_M$.

For example, `the ten` is definite. If $|A| \neq 10$, $(\textit{the ten}^A)_M$ is empty, i.e. $\textit{the ten}_M(A, B)$ is false for all $B \subseteq M$. If there are just five boys (in the discourse universe), then `the ten boys` doesn't refer to anything. But if $|A| = 10$, $(\textit{the ten}^A)_M$ is generated by A. And `Mary's books` is definite (on the universal interpretation; see the next section): it refers whenever Mary 'possesses' at least one book, and then it refers to the set of books she 'possesses'. We get reference to single objects as reference to singletons: I_a is definite, and if $a \in M$, $(I_a)_M$ is generated by $\{a\}$.

But `every`, `all` are not definite, which seems according to intuition: `all students` doesn't refer to the set of students any more than `three students` or `most students` does. The reason is that $\textit{all}(\emptyset, B)$ is true for every B, so for $A = \emptyset$, there is no *non-empty* set generating $(Q^A)_M = \mathcal{P}(M)$.[23]

If we distinguish different notions of definiteness, we should expect them to overlap but not coincide. For example, some NPs are semantically indefinite but morphosyntactically definite:

(56) the inventor of a new drug, the manuscripts of some professors

(These are sometimes called *weak definites*; see Poesio (1994).) The difference between pragmatic and semantic definiteness seems to run deeper, in that it involves the very notion of meaning appealed to. If one thinks that the meaning of `the ten boys` or `Mary's books` *in itself* carries a familiarity condition, then that is not captured in the GQ account of Dets and NPs. But the semantic notion is unique in that it has (a) a clear intuitive content in terms of reference, and (b) a precise model-theoretic counterpart.[24]

[23] If we define *all with existential import* as the quantifier $\textit{all}_{ei}(A, B)$ iff $\emptyset \neq A \subseteq B$, then $\textit{all}_{ei}$ is definite. Many linguists seem to think that `all` means $\textit{all}_{ei}$ in English; I prefer to regard the fact that it is often *odd to say* `All A are B`, when (we know that) A is empty, as pragmatic rather than semantic. And the conclusion that `all A` is definite, in the semantic sense of referring to A, seems rather unwelcome.

[24] So one would think that the semantic notion would be fairly undisputed, but in fact it is often criticized by linguists; see e.g. Abbott (2004). Most of this

0.10 Possessives

I end by considering how GQ theory can be applied to the semantics of the rich and productive class of *possessive NPs*, exemplified by

(57) a. Mary's books, my sisters
b. several students' bicycles, each woman's parents
c. most of Mary's books, none of my sisters
d. two of most students' term papers, exactly one of each woman's parents[25]

Here (57a) and (57b) consist of a *possessor NP* + `'s` followed by a noun, so we can take [NP 's] to be a *possessive determiner*, interpretable as a (Conserv and Ext, but not Isom) type $\langle 1, 1\rangle$ quantifier. Possessive NPs denote type $\langle 1\rangle$ quantifiers, just as other NPs. In (57c) and (57d) they are preceded by [Det of], and the whole phrase can be seen as a possessive NP.[26]

However, there is no standard application of GQs to the semantics of possessives. The vast linguistic literature on possessives mostly focuses on a small portion of the full class of possessive NPs, like those in

(58) Mary's brother, John's portrait, my book, the table's leg, *the leg's table, God's love

criticism, however, in fact concerns Barwise and Cooper's treatment of existential there sentences in terms of their notions of strong and weak Dets. That treatment can indeed be disputed, as Keenan (1987) did, but is an entirely different matter (Keenan had no objections to their concept of definiteness). We already found reasons to doubt that unacceptability in existential there sentences amounts to definiteness. For example, most theorists regard `all A` as non-definite, in any of the senses we have mentioned, but it is unacceptable in such sentences. Indeed, `all A` is strong in Barwise and Cooper's sense, but not semantically definite. A more pertinent criticism concerns the fact (recognized by Barwise and Cooper) that `both` and `the two` denote the same generalized quantifier, but only `the two` can occur in partitives: `one of the two/*both men`. Ladusaw (1982) concludes that `both` is not definite, arguing that it does not refer to a unique group. In any case, this single example apart, Barwise and Cooper's definition covers a vast number of Dets (and NPs) that are indisputably semantically definite.

[25] This section sketches parts of the account of the semantics of possessive in Peters and Westerståhl (2012).

[26] Many theorists would not call the expressions in (57c) and (57d) possessive but *partitive*. But given that the NP in a partitive of the form [Det of NP] is supposed to be definite (cf. footnote [24]), we see directly that the NPs in (57d) are not partitive. Still, they are perfectly meaningful, and their meaning is given by the semantics for possessive NPs sketched here.

where the possessor NP is a proper name or a pronoun or a simple singular definite, and where the noun is in the singular.[27] For these, one might think, GQ theory is of little relevance. But, (a) if possible, a uniform semantics for the whole spectrum of possessive Dets and NPs would surely be preferable, and (b) the study of the various quantified cases turns out to shed light also on the simpler ones. GQ theory does have good things to offer the analysis of possessives, only they are not yet in the semanticist's standard toolbox.

'Possessiveness' is here taken to consist in a certain syntactic form with a certain kind of meaning. This may seem somewhat stipulative, but in fact the class thus delineated is quite natural. It means, however, that some things often called 'possessive' are not included. One example is constructions with `have, belong, own`:

(59) a. Many paintings that John has are quite valuable.
b. Three books belonging to Mary are lying on the sideboard.

These are quite ordinary constructions with a transitive verb, that happens to indicate ownership or some similar relation. Apart from that, they are not different from, say,

(60) a. Many paintings that John bought are quite valuable.
b. Three books written by Mary are lying on the sideboard.

Possessive NPs always involve a *possessive relation*, holding between *possessors* and *possessions*. But not much hangs on which particular relation it happens to be. In particular, it need not have anything to do with ownership or possession. For example, `Mary's books` can be the books she owns, or bought, or read, or wrote, or designed, or is standing on to reach the upper shelf, etc. Often the possessive relation comes from a relational noun, as in `Henry's sisters, my students, Susan's enemies`; again nothing to do with possession in the literal sense. But in (59) the characteristic syntax of possessives is missing. Note, by the way, that in this case there are logically equivalent sentences which do involve possessive NPs:

(61) a. Many of John's paintings are quite valuable.
b. Three of Mary's books are lying on the sideboard.

[27] The main exceptions are Keenan and Stavi (1986) and Barker (1995). Keenan and Stavi were the first to emphasize the wide variety of possessive Dets. Barker gave a systematic treatment of a subclass (those of the form exemplified in (57a) and (57b)), related to but distinct from the semantics proposed here.

But that doesn't make (59) possessive.

Freedom of the possessive relation is one of the characteristic features of possessives. The freedom involved in cases like `Mary's books` is well-known, but the possessive relation is free also with relational nouns. That is, although it often comes from the relational noun (as the *inverse* of the relation expressed by the noun), it doesn't have to. `Henry's sisters` can be the sisters (of someone else) that he was selected to escort. `Ruth's mothers` usually *has to* involve another relation than the inverse of 'mother-of'. The point is, however common a default selection of the possessive relation might be, there are *always* circumstances under which a different relation, coming from the context and unrelated to the linguistic co-text, is chosen.[28]

Perhaps the most important characteristic of possessives is that they always *quantify over possessions*. Often the quantification is universal, but it can also be existential. Consider

(62) a. The teacher confiscated three children's paint sprayers.
b. The teacher confiscated three children's paint sprayers hidden around in bushed near the school.

(63) a. Mary's dogs escaped.
b. When Mary's dogs escape, her neighbors usually bring them back.

The most natural reading of (62a) has the teacher confiscating *every* paint sprayer of each of the three children. But in (62b) the most plausible reading is existential. There are three children who had paint sprayers and for whom the teacher discovered at least *some* of the child's paint sprayers in the bushes. Similarly for the sentences in (63). Other sentences are ambiguous:

(64) Most cars' tires were slashed.

has a reading where there is a set with more than half of the (salient) cars such that every car in this set has all its tires slashed, and another, perhaps more natural, where each of these cars had some tire slashed.

Now, these readings can be made explicit as follows:

[28] This criterion allows us to distinguish gerundive nominals, as in

(i) John's not remembering her name annoyed Mary.

from possessives: here the relation is completely fixed.

(65) a. All of Mary's dogs escaped.
b. When some of Mary's dogs escape, her neighbors usually bring them back.

Furthermore, when the quantification over possessions is explicit, it can be (almost) *any* quantifier:

(66) a. Most of Mary's dogs escaped.
b. When two of Mary's dogs escape, her neighbors usually bring them back.

These contain possessive NPs as in (57c) and (57d). Whether implicit or explicit, quantification over possessions is always there.

Since possessor NPs can also be quantified, many possessive NPs (perhaps all, if you treat e.g. proper names as quantifiers) are *doubly quantified*. This has several consequences that GQ theory helps illuminate. To state some of them, it is convenient to spell out uniform truth conditions for sentences involving possessive NPs. In the most general case such a sentence has the form

(67) $(Q_2$ of$)$ Q_1 C's A are B

Here Q_2 quantifies over possessions; it may be implicit but Q_2 is always part of the semantics. Q_1 quantifies over possessors. We define an operation *Poss*, taking two type $\langle 1,1\rangle$ quantifiers, a set C, and a binary relation R (the possessive relation) as arguments, and returning a type $\langle 1,1\rangle$ quantifier as value. To state the definition, the following abbreviations are convenient. First, for any a,

$$R_a = \{b\colon R(a,b)\}$$

is the set of things 'possessed' by a. Second, for any set A,

$$dom_A(R) = \{a\colon A \cap R_a \neq \emptyset\}$$

is the set of a which 'possess' things in A. Now define:[29]

(68) $$Poss(Q_1,C,Q_2,R)(A,B) \Leftrightarrow Q_1(C \cap dom_A(R), \{a\colon Q_2(A \cap R_a, B)\})$$

[29] I suppress the universe M; this is harmless, since one can show that if Q_1 and Q_2 are Conserv and Ext, so is $Poss(Q_1, C, Q_2, R)$.

(68) expresses the truth conditions of (67).[30] Indeed, in a compositional analysis, one may take *Poss* to be the denotation of the possessive morpheme `'s`, and $Poss(Q_1,C,Q_2,R)^A$ as the denotation of the possessive NP. When Q_2 is implicit, it has to be provided by the context. And the idea of freedom entails that for all sentences of the form (67), there are situations where R has to be supplied by context, even if A comes from a relational noun.[31] For example, applying (68) to

(69) Some of at least three students' library books are overdue.

this sentence says that at least three (Q_1) students (C) who borrowed (R, the most natural possessive relation here) library books (A) are such that at least one (Q_2) of the library books they had borrowed is overdue (B). This seems correct.

Notice that the phrase `who borrowed library books`, which comes from the restriction of C to $dom_R(A)$ in (68), is unnecessary: the truth conditions in this case are

$$|\{a \in C\colon A \cap R_a \cap B \neq \emptyset\}| \geq 3$$

so whether (69) quantifies over students who borrowed library books or students in general is irrelevant. But this is not always the case. Consider

(70) Most people's grandchildren love them.

This is probably true, but note that most people in the world don't have any grandchildren (they are too young for that). But this fact has nothing to do with the truth value of (70). The quantifier *most* (Q_1) indeed quantifies only over 'possessors', i.e. over people who have grandchildren, saying that most of *these* are such that their grandchildren love them. This is why (68) in general *narrows* C to $dom_A(R)$.

[30] We also have to account for the semantics of non-quantified possessive Dets such as `Mary's`. This can be done by decomposing the quantifier I_m as $all^{\{m\}}_{ei}$ and then applying (68). This gives the right truth conditions (for the universal reading), but one problem is that the decomposition is not unique: we also have $I_m = all^{\{m\}}$, and this decomposition gives `Mary's friends are nice` rather odd truth conditions: either Mary has no friends or all of her friends are nice. It also raises the question of whether all non-quantified possessive NPs are decomposable. Both issues are dealt with at length in Peters and Westerståhl (2012).

[31] If A comes from, say, `parent`, A is the set of parents, i.e. the set of individuals standing in the parent-of relation to something, and R is the *inverse* of the parent-of relation.

The term *narrowing* is from Barker (1995), who argued that narrowing is always in place. I tend to agree, though counter-examples have been suggested. It can be shown that for *symmetric* Q_1, as in (69), narrowing has no effect, which explains why its presence is often not felt. But for non-symmetric Dets like `every` and `most`, it has a clear semantic effect. For example, without narrowing, (70), would be made trivially false (with $Q_2 = all$) by the fact that most people have no grandchildren.[32]

Here is an application of the uniform truth conditions. In the literature, possessiveness is usually tied to *definiteness*. As Abbott (2004) says, "Possessive determiners ... are almost universally held to be definite" (p. 123). A more nuanced view, originating with Jackendoff, is put forth in Barker (2011): possessive NPs *inherit* definiteness from their possessor NPs. Since we have precise truth conditions for possessives as well as definites (section 0.9), we can find out what the facts are. Peters and Westerståhl (2006) (ch. 7.11) prove the following.

Fact 0.6 *If Q_1 is semantically definite and Q_2 is universal, then, for all C and R, $Poss(Q_1,C,Q_2,R)$ is semantically definite. Also, in practically all cases when Q_1 is not definite, or Q_2 is not universal, $Poss(Q_1,C,Q_2,R)$ is not definite.*

So we see that in general, there is no reason to expect possessive Dets or NPs to be semanticlly definite. Even a simple phrase like `Mary's dogs` is only definite under the universal reading. Further, Fact 0.6 shows what is right about the Jackendoff/Barker inheritance claim, but also what is wrong: the definiteness of the possessor NP is not inherited when quantification over possessions is not universal. Consider again

(63b) When Mary's dogs escape, her neighbors usually bring them back.

Here `Mary's dogs` is not semantically definite: the possessive NP doesn't refer to any particular set of dogs.

Another illustration of the potential of model-theoretic semantics to clarify important issues concerning the semantics of possessives is af-

[32] Peters and Westerståhl (2006) show that the correct non-narrowed version of the truth conditions are

(i) $Poss^w(Q_1,C,Q_2,R)(A,B) \Leftrightarrow Q_1(C, \{a\colon A \cap R_a \neq \emptyset \;\&\; Q_2(A \cap R_a,B)\})$

So if most C are such that $A \cap R_a$ is empty, (i) makes (70) false with $Q_1 = most$ and $Q_2 = all$. And it doesn't help to let $Q_2 = all_{ei}$ (see footnote [23]), for then (70) will entail that most people have grandchildren, which is equally absurd.

forded by the monotonicity behavior of possessive Dets and NPs. This behavior is quite interesting, but to study it one needs precise truth conditions like those in (68); see Peters and Westerståhl (2006), ch. 7.12. I will end, however, with a different illustration: the meaning of *negated* sentences with possessive NPs.

To begin, inner and outer negation (section 0.6) applies to possessive Dets just as to all other Dets, and it is easy to check that the following holds:

Fact 0.7

(a) $\neg Poss(Q_1, C, Q_2, R) = Poss(\neg Q_1, C, Q_2, R)$
(b) $Poss(Q_1, C, Q_2, R)\neg = Poss(Q_1, C, Q_2\neg, R)$
(c) $Poss(Q_1\neg, C, \neg Q_2, R) = Poss(Q_1, C, Q_2, R)$

For example,

(71) Not everyone's needs can be satisfied with standard products.

seems to be a case of outer negation: it says that at least someone's needs cannot be thus satisfied. On the other hand, consider

(72) Mary's sisters didn't show up at the reception.

Here Q_2 is naturally taken as universal, and the sentence says that none of the sisters showed up; this is inner negation ($all\neg = no$).

However, if we make Q_2 explicit,

(73) All of Mary's sisters didn't show up at the reception.

there is another interpretation, which is neither outer nor inner negation, namely, that not all the sisters showed up (but some of them may have). This possibility is not covered in Fact 0.7. But given that we have *two* quantifiers and *two* types of negation, the combination is natural enough. Let us call it *middle negation*:

$$\neg^m Poss(Q_1, C, Q_2, R) =_{\text{def}} Poss(Q_1, C, \neg Q_2, R)$$

It follows from Fact 0.7 (c) that $\neg^m Poss(Q_1, C, Q_2, R)$ is also equal to $Poss(Q_1\neg, C, Q_2, R)$. Westerståhl (2012) shows that possessive Dets, in view of these three types of negation, span a *cube*, rather than a square, of opposition. There is much more to say on this subject, but I leave it here, noting only that the study of how negation interacts with possessives would hardly have been possible without the GQ tools at our disposal.

0.11 In conclusion

The purpose of this chapter has been to illustrate how tools from generalized quantifiers in model theory can contribute to our understanding of the semantics of quantification in natural languages, which in English is carried out mostly by means of determiners and noun phrases. I have chosen, except perhaps in the preceding section, tools that are by now more or less standard in formal semantics. There are many applications of GQ theory to natural language quantification I did not touch upon, most notably the use of polyadic quantifiers, for example in reciprocal sentences, and questions concerning the logical expressive power of various quantifiers. And I said nothing about processing quantified expressions.

Note also that I have not been trying to convey the impression that GQ theory can account for every feature of natural language quantification. So far it has little to say, for example, about the inner composition of determiners (treating, for example, `more than two-thirds of the` as an unanalyzed Det), or about meaning distinctions 'below' the level of generalized quantifiers.[33] But I hope that the illustrations given here provide some feeling for the amazing ease with which the tools of GQ theory, invented for mathematical logic purposes, can be used for the semantics of ordinary language.

[33] Such as the distinction between `both` and `the two` (footnote [24]), or between `at least two` and `more than one`: cf. (Hackl, 2000)

(i) a. At least two men shook hands.
    b. #More than one man shook hands.

Szabolcsi (2010) presents a number of such challenges for GQ theory, or for going beyond (without abandoning) GQ theory in the linguistic analysis of quantification.

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