Negation and Quantification.
A New Look at the Square of Opposition

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1 Outline

This paper is about how the fundamental logical notions of negation and quantification interact. The subject is as old as logic itself, and appears in various forms in all ancient traditions in logic, but perhaps most explicitly and systematically in the Western tradition starting with Aristotle, which is the one I will refer to here. Many of the issues raised by Aristotle and his followers are still subject of lively debate among linguists and philosophers today.

Negation is closely tied to *opposition*: a statement and its negation are in some sense opposed to each other. There are different kinds of opposition (and philosophers love displaying all kinds of opposing pairs), but Aristotle's main distinction is between *contradictory* and *contrary* opposition or negation. The contradictory negation of *Socrates is wise* is simply *Socrates is not wise*, which is true if and only if the positive statement is false. But there are several contrary negations: *Socrates is non-wise/unwise/foolish/...*; the hallmark of contrariety is that a statement and its contrary cannot both be true.

The contrast between contrariety and contradictoriness becomes clearer, and simpler, when applied to quantified statements. Aristotle's syllogistics—the beginning of modern logic—is a systematic study of relations between the four quantifiers *all, no, some, not all*, and in particular of how they interact with negation. The contrary of *all* is *no*, whose contradictory negation is *some*. (Really it is the corresponding statements, *all A are B*, *no A are B*, etc. that have these relations, but they are easily lifted to the quantifiers themselves.) Aristotle's claims about these matters were later summarized diagrammatically and extended in the famous Square of Opposition (Fig. 1).

This square is not only an elegant way of displaying logical relations, but is intended to provide important information about the meaning of negation and quantification in natural languages. Among questions related to it that have been, and still are, intensely debated are the following (see Horn (1989) for a rich account of the history of negation):

(a) What (if anything) is less valuable/real/natural/informative about negative statements as compared to positive ones?

(b) What (if anything) makes a statement negative rather than positive?

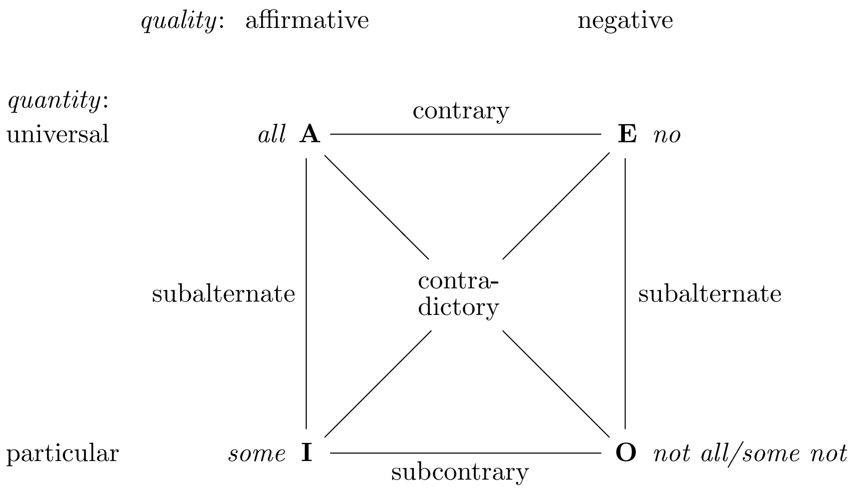

Figure 1: The Aristotelian Square

(c) What sort of thing or state of affairs makes a negative statement *true*?

(d) Why is the **O** corner never lexically realized (in any language)?

(e) Does the **A** corner have *existential import*, as in the Aristotelian square, i.e. does *All A are B* entail (or presuppose) that there are *A*s? What about the other corners?

For example, Aristotle (presumably), most (but not all) medieval logicians, and most (but not all) modern linguists claim that *All A are B* indeed entails (or presupposes) that there are *A*s. By contrast, universal quantification for most modern logicians since Frege does *not* have existential import.

This paper only indirectly addresses the above questions (though I will be glad to talk about them in discussion). Instead, I focus on another issue, also intimately related to the meaning of negation and quantification in natural languages, namely, what exactly are the relations along the *sides* of the square? My claim is that it is misleading to think of contrariety, subalternation, etc. here; rather, the 'horizontal' relations are one and the same: *inner negation* or *post-complement*; and the 'vertical' ones are *dual*, as in Fig. 2.

The main argument is that this Modern Square—but not the Aristotelian Square—applies to all kinds of natural language quantification, not just to the Aristotelian quantifiers. Indeed, those four quantifiers instantiate a very general pattern, with examples such as *at least three, exactly six, all but five, most, many, few, more than two-thirds of the, the seven, Mary's, some students'*, . . . , each one related to its negations and dual according to the modern square. Nothing similar works for the classical square. Here is an outline of the paper.

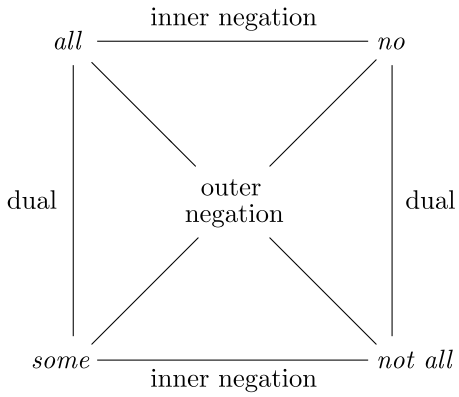

Figure 2: The modern version of Aristotle's square

1.1 Generalized quantifiers

(Binary) *generalized quantifiers* are relations between subsets of the universe. Thus $all(A, B)$ says that A is a subset of B, $no(A, B)$ that the intersection of A and B is empty, $some(A, B)$ that it is non-empty, *at least three*(A, B) that it has at least three elements, $most(A, B)$ that it contains more elements than the difference $A - B$ does, *the seven*(A, B) that there are (exactly) seven As and that they are all B, *some student's*(A, B) that some student 'possesses' As and all the As 'possessed' by her/him are B, etc. In English and many other languages, binary quantifiers interpret determiners. In languages with so-called A quantification, they interpret adverbs, auxiliaries, or other devices for quantification.

For each such quantifier Q we can define its outer (contradictory) negation: $\neg Q(A, B)$ iff not $Q(A, B)$; its inner negation: $Q\neg(A, B)$ iff $Q(A, A - B)$; and its dual: $Q^{\mathrm{d}} = \neg(Q\neg)$. So each such Q spans a (modern) square:

$$square(Q) = \{Q, Q\neg, \neg Q, Q^{\mathrm{d}}\}$$

One verifies that $square(Q)$ has either 4 elements (the normal case) or 2 (it can happen that $Q = Q\neg$), and that each element in $square(Q)$ spans the same square. (For an overview of generalized quantifiers in language and logic, see Peters and Westerståhl (2006).)

1.2 Classical vs. modern squares

Aristotle's square concerns just the four classical quantifiers. Trying to extend it to other quantifiers yields the following definition: A *classical square* is an arrangement of four quantifiers as traditionally ordered and with the same logical relations—contradictories, contraries, subcontraries, and subalternates—between the respective positions. Here two statements are subcontraries if they cannot both be false, and a statement is subalternate to another statement if it is implied by it.

However, the result is that, for example, the square

[**A**: *at least five*; **E**: *no*; **I**: *some*; **O**: *at most four*]

is classical. This looks very unnatural. There is no interesting sense, it seems, in which *no* is a negation of *at least five* or *at most four.* The classical Square of Opposition simply doesn't generalize in a natural way to other quantifiers.

1.3 Identifying the corners

Medieval logicians classified the corners of the square by means of the categories *quantity*: universal (*all, no*) / particular (*some, some not*) and *quality*: affirmative (*all, some*) / negative (*no, not all*). These properties are often hard to apply to other quantifiers. For example, *no fewer than five* may seem negative, and *more than four* positive. But extensionally they are the same quantifier!

There are, however, clearcut—and Aristotelian—semantic properties that can often be used to at least partly identify the corners. One is *monotonicity*; in fact, the syllogisms describe exactly the monotonicity behavior of the four Aristotelian quantifiers, and these properties apply to other quantifiers as well. Another is *symmetry*; Aristotle noted that the quantifiers in the **I** and **E** corner are symmetric (though he didn't use these terms): $Q(A, B)$ implies $Q(B, A)$. Using these properties, and the further criterion that, if possible, that the standard *square*(*all*) should be a *special case* of *square*(Q), one can in most common cases classify the corners of *square*(Q).

1.4 Case studies

1.4.1 Numerical quantifiers

These are Boolean combinations of quantifiers of the form *at least n*, such as *at most six* and *exactly five.* Their squares—i.e. their patterns of negation—are interesting, in terms of how they are realized in English, and how little the classical approach says about them.

1.4.2 Proportional quantifiers

Typical proportional quantifiers are *most* and *at least two-thirds of the*; the other quantifiers in their squares also have natural renderings as English determiners. It is notable that—except for *most*!—the definite article is required in English. This reopens the issue of existential import (e.g. is it presupposed or entailed?), but with more clearcut examples than those in the traditional square.

1.4.3 Exception quantifiers

In *No student except Mary came to the opening*, the phrase *no _ except Mary* can be seen as a quantifier: *no _ except Mary*(A, B) iff $A \cap B = \{\text{Mary}\}$, with a corresponding square of opposition.

1.4.4 Possessive quantifiers

Possessive quantifiers like *Mary's, two students', several of most teachers'*, have interesting logical and semantic properties, also as regards their interaction with negation. (See Peters and Westerståhl (2013) for much more on the semantics of possessives.) A main observation is that they involve *two* quantifiers, one over possessors and one over possessions (relative to some 'possessive' relation, that need not have anything to do with ownership). In *No car's tires were slashed*, Q_1 = *no* quantifies over possessors (cars), but the quantier Q_2 over possessions (tires) is implicit; presumably in this case it is *some.* In *At least one of most teachers' pupils failed the exam*, both Q_1 (*most*) and Q_2 (*at least one*) are explicit.

This allows for several ways in which negation can apply. In addition to standard outer and inner negation, we also find what we call *middle negation.* For the sentence *All of Mary's friends didn't come to the party*, the outer negation reading is implausible (but it applies in other cases), and the inner negation reading says that none of Mary's friends showed up. This is a possible reading, but more likely is the reading that not all of them showed up (though some of them may have): middle negation. The result is that there are two kinds of squares for possessive quantifiers. For example, the 'middle' square of *Mary's* more accurately reflects how *Mary's* behaves under negation than the ordinary *square*(*Mary's*). However, I show that the full (logically possible) behavior of possessives under negation can be represented in a *cube* of opposition.

2 Main claims

- It may seem—and it is usually assumed—that the only difference between the Aristotelian square of opposition and the modern *square*(*all*) is that *all* has existential import in the former but not the latter square. This impression is mistaken. The main difference concerns the relations along the sides of the squares. That difference is hardly visible for *square*(*all*), but it becomes evident for the squares spanned by other quantifiers.

- The issue of existential import for *all* is real, and alive especially among linguists. However, it does not really concern the properties of the square, i.e. the interaction between quantification and negation (except that if *all* has existential import, its contradictory negation becomes a rather unnatural quantifier). On the other hand, the square highlights other issues about existential import, e.g. for the proportional quantifiers.

- The fact that 'modern' squares—but not traditional ones—are generated by any binary quantifier, and hence by any English determiner denotation, constitutes a main argument that they capture important facts about negation and quantification in natural languages.

- The traditional way of identifying the corners of the square in terms of quantity and quality are not sufficiently precise to be applicable to other quantifiers. In particular, ‘quality’ relies on identifying *negative* statements (or quantifiers), but it is very doubtful that this can be done in a coherent way. However, the criteria can be replaced by relying on precise semantic—and still very much Aristotelian!—properties of the quantifiers involved. In particular, the monotonicity behavior of quantified expression is an area where an exact study of semantics is both possible and rewarding. As to the square, a question that remains after classification is: What (if any) is the significance of the various corners of the square?

- Investigating the (modern) squares of various quantifiers is a fruitful way to study the interaction between negation and quantification in natural languages. Many questions arise that have no counterpart for the Aristotelian Square. In particular, possessives provide a rich field of application for generalized quantifier theory to natural language. Among other things, they exhibit a new form of negation, distinct from outer and inner negation, and a full description of how possessives interact with negation turns out to require a *cube* of opposition.

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