Midpoints

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Introduction

A *midpoint* is a quantifier identical to its own postcomplement, i.e. a fixed point of the postcomplement operation. I'm borrowing the term from Ed Keenan, who noticed that such fixed points lead to curious logical equivalences, like the one between (1-a) and (1-b):

(1) a. Between one-third and two-thirds of the students passed the exam.
  b. Between one-third and two-thirds of the students didn't pass the exam.

Keenan, however, did not use "midpoint" in this way, but rather for a feature of a certain class of proportional fixed points (section 2 below). But there are many other examples, and for lack of a natural descriptive name, I shall here appropriate the label *midpoint* for any such fixed point.

Keenan discovered that, far from being an anomaly, midpoints exist in great numbers. He proved theorems about them and gave numerous English examples; see Keenan (2005, 2008). In this note I take a new look at these results and their proofs.

A secondary aim is to illustrate the difference between two approaches to semantics: *global* and *local*. Like most linguists, Keenan usually prefers a local perspective: fix a universe M of individuals and consider predicates, relations, functions, quantifiers, and other higher-type objects over M. He then observes facts like the following: if Q and Q' are midpoints, so is $Q \vee Q'$. Literally, this result quantifies over *all sets of subsets of* M (all type $\langle 1 \rangle$ quantifiers on M). However, in this case, the same proof works *for every universe* M, so in effect, you are quantifying over M too. This is the logician's global perspective.

A global result implies the corresponding local version, but the converse can fail, although it didn't in the example just mentioned. *Definability* results provide the clearest examples. Keenan and Stavi (1986) proved that all type $\langle 1 \rangle$ quantifiers on a finite universe M are definable as Boolean combinations of Montagovian individuals (and hence—in a liberal sense—denotable by English DPs). But the defining sentence depends on $|M|$ ($|X|$ is the cardinality of X), and there is no global version of this theorem. A global definability result requires the same defining formula for every universe.[1]

[1] An example: the type $\langle 1,1,1 \rangle$ quantifier *more than*, defined, for all M and all $A,B,C \subseteq M$, by

(i) $\textit{more than}_M(A,B,C) \Leftrightarrow |A \cap C| > |B \cap C|$,

as in "More men than women smoke", turns out to also be definable in terms of the two type $\langle 1,1 \rangle$ quantifiers *most* and *infinitely many*; see e.g. Peters and Westerståhl (2006), ch. 13.2.

My point about midpoints will be that the global perspective gives a better view of the issues involved and simplifies proofs. But this doesn't mean that it is always preferable. An interesting contrast is provided by a question dual to the one about midpoints: the existence of *self-dual* quantifiers. I point out that here the global approach is barren, but the local perspective provides some linguistic insights.

1 Preliminaries

1.1 Quantifiers

A (global) *type* $\langle 1,1\rangle$ (generalized) *quantifier* Q associates with each non-empty set M a (local) *type* $\langle 1,1\rangle$ *quantifier* Q_M *on* M, i.e. a binary relation between subsets of M.[2] Similarly for a type $\langle 1\rangle$ quantifier, associating with each M a set of subsets of M. When a type $\langle 1,1\rangle$ quantifier interprets an English Det, we use that Det to name it:

(2) a. $\mathit{all}_M(A,B) \Leftrightarrow A \subseteq B$
b. $\mathit{exactly\ five}_M(A,B) \Leftrightarrow |A\cap B| = 5$
c. $\mathit{most}_M(A,B) \Leftrightarrow |A\cap B| > |A-B|$
d. $\mathit{infinitely\ many}_M(A,B) \Leftrightarrow A\cap B|$ is infinite
e. $\mathit{between\ one\text{-}third\ and\ two\text{-}thirds\ of\ the}_M(A,B) \Leftrightarrow 1/3 \leq |A\cap B|/|A| \leq 2/3$ [3]

Recall that Det denotations have the properties of *conservativity* and *extension*:

(CONSERV) $Q_M(A,B) \Leftrightarrow Q_M(A,A\cap B)$ (all M, all $A,B\subseteq M$)

(EXT) If $A,B\subseteq M\subseteq M'$, then $Q_M(A,B) \Leftrightarrow Q_{M'}(A,B)$

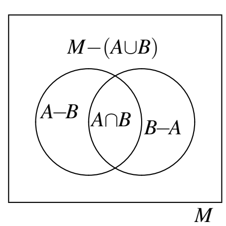

Figure 1: The four sets relevant to a type $\langle 1,1\rangle$ quantifier on M

CONSERV says that $B-A$ doesn't matter for the truth value of $Q_M(A,B)$, EXT says that $M-(A\cup B)$ doesn't matter; together they restrict the domain of quantification to A. Many

[2] I give only the bare essentials of the generalized quantifier framework used in this note; for more details, examples, and explanations, see any overview of GQ theory, such as Peters and Westerståhl (2006), Keenan and Westerståhl (2011), Westerståhl (2011). Keenan treats quantifiers as *functions* rather than relations; then a type $\langle 1\rangle$ (or, if you will, type $\langle\langle e,t\rangle,t\rangle$) quantifier is, on each M, a function from subsets of M to truth values, and a type $\langle 1,1\rangle$ (or $\langle\langle e,t\rangle,\langle\langle e,t\rangle,t\rangle\rangle$) quantifier is a function from subsets of M to type $\langle 1\rangle$ quantifiers on M. For present purposes, this is just a notational variant of the relational approach.

[3] Keenan adds the condition $A\neq\emptyset$ on the right-hand side, which seems right in view of the obligatory presence of the definite article in the corresponding Det. For simplicity, I leave out that condition here.

Det denotations (e.g. all those in (2)) also satisfy ISOM, which says that only the *cardinalities* of the relevant sets matter (in general, all four partition sets in Fig. 1; under CONSERV and EXT, just $|A-B|$ and $|A\cap B|$ matter).

EXT also entails that we often can drop the subscript $_M$. In a way this hides the global/local distinction, but note that quantifiers are essentially global objects, with a local version on each universe—the condition EXT cannot even be formulated from a strictly local perspective. From now on, unless otherwise noted, type $\langle 1,1\rangle$ quantifiers are assumed to be CONSERV and EXT.

1.2 Boolean operations

Standard Boolean operations apply directly to quantifiers:

(3) Definition:

a. $\neg Q(A,B) \Leftrightarrow$ not $Q(A,B)$
b. $(Q\wedge Q')(A,B) \Leftrightarrow Q(A,B)$ and $Q'(A,B)$
c. $(Q\vee Q')(A,B) \Leftrightarrow Q(A,B)$ or $Q'(A,B)$

$\neg Q$ is the *outer negation* of Q. But quantifiers have two other kinds of negation: an *inner negation* $Q\neg$, that Keenan calls the *postcomplement* of Q, and a double, inner-outer (or vice versa), negation Q^d, called the *dual* of Q:

(4) Definition:

a. $Q\neg(A,B) \Leftrightarrow Q(A,A-B)$
b. $Q^d = (\neg Q)\neg = \neg(Q\neg)$

Then

$$square(Q) = \{Q, Q\neg, \neg Q, Q^d\}$$

is a modern version of the Aristotelian *square of opposition*, generalized to any quantifier Q.[4] That it makes sense to say that any Q spans a unique square follows from:

(5) If $Q' \in square(Q)$, then $square(Q') = square(Q)$.

The following facts are easy to establish:

(6)
a. The three negations are idempotent, i.e. $Q = \neg\neg Q = Q\neg\neg = Q^{dd}$.
b. $\neg(Q\wedge Q') = \neg Q \vee \neg Q'$ and $\neg(Q\vee Q') = \neg Q\wedge\neg Q'$ (de Morgan laws)
c. $(Q\wedge Q')\neg = Q\neg\wedge Q'\neg$ and $(Q\vee Q')\neg = Q\neg\vee Q'\neg$
d. $(Q\wedge Q')^d = Q^d \vee Q'^d$ and $(Q\vee Q')^d = Q^d\wedge Q'^d$

Since a quantifier is always distinct from its outer negation, if follows that $square(Q)$ has either 4 or 2 members. So in principle there are just two ways for a $square(Q)$ to be ‘degenerate’: it contains either a midpoint or a self-dual quantifier:

[4] For an account of the (considerable) differences between the traditional and the modern square, and a study of $square(Q)$ for various Q, see Westerståhl (2012).

(7) Definition:

a. Q is a *midpoint* if $Q = Q\neg$
b. Q is *self-dual* if $Q = Q^d$

I mentioned a midpoint in the Introduction: the equivalence of (1-a) and (1-b) shows that *between one-third and two-thirds of the* = (*between one-third and two-thirds of the*)¬. I gave no example of a self-dual quantifier; we will see why presently.

1.3 The number triangle

We often restrict attention (as Keenan usually does) to *finite universes*; this is marked Fin. It then follows from the definitions above that under Fin, a type $\langle 1,1\rangle$ Conserv, Ext, and Isom quantifier Q can be identified with a *binary relation between natural numbers*. More precisely, using the same name for this relation, define

(8) $Q(k,m) \Leftrightarrow$ for some A,B with $|A-B| = k$ and $|A \cap B| = m,\ Q(A,B)$ [5]

For example,

(9) a. $all(k,m) \Leftrightarrow k = 0$
b. $exactly\ five(k,m) \Leftrightarrow m = 5$
c. $most(k,m) \Leftrightarrow m > k$
d. *between one-third and two-thirds of the*$(k,m) \Leftrightarrow 1/3 \leq m/(k+m) \leq 2/3$

The *number triangle* is just $\mathbb{N}^2$ turned 45 degrees; see Fig. 2. So a quantifier Q is simply

(0,0)
(1,0) (0,1)
(2,0) (1,1) (0,2)
(3,0) (2,1) (1,2) (0,3)
(4,0) (3,1) (2,2) (1,3) (0,4)
. . . .

Figure 2: The number triangle

an area in the number triangle; Fig. 3 gives examples. Johan van Benthem realized early on that this visual representation of (Conserv, Ext, and Isom) quantifiers is an enormously useful tool for finding properties of and proving facts about them (under Fin); see (van Benthem 1984). I will make essential use of it below.

[5] Note that the *first* argument of the relation is $|A-B|$ and the *second* is $|A \cap B|$. This is purely conventional.

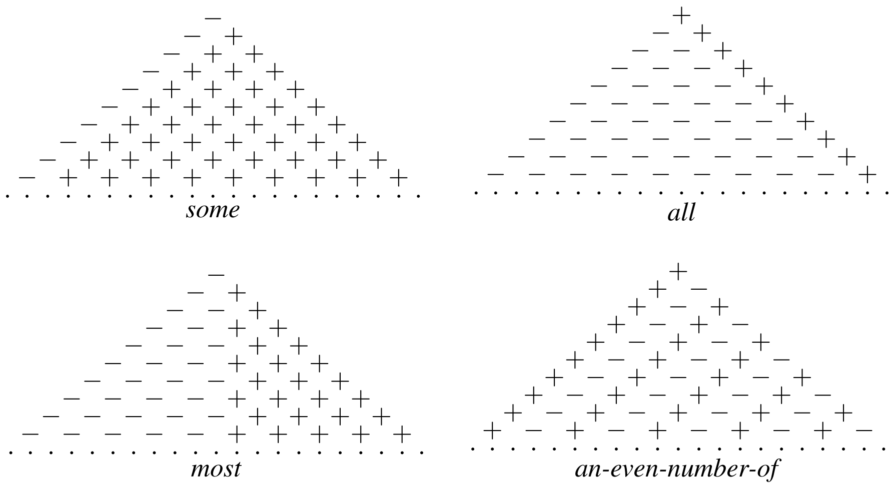

Figure 3: Some quantifiers in the number triangle

2 Keenan on midpoints

The perspective in Keenan (2005, 2008) is local: a finite universe M, say $|M| = n$, is fixed, and a midpoint is a local quantifier Q_M on M such that $Q_M = (Q_M)\neg$. Keenan (2005) considers type $\langle 1 \rangle$ quantifiers: the three kinds of negation, and thus the notion of a midpoint, transfer in an obvious way to such quantifiers. One result counts the number of type $\langle 1 \rangle$ midpoints on M: without ISOM it is

$$2^{2^{n-1}}$$

and restricted to ISOM quantifiers it is

$$2^{(n+1)/2} \text{ if } n \text{ is odd, and } 2^{(n+2)/2} \text{ if } n \text{ is even.}$$ [6]

Thus, on a 5 element universe there are $2^{16} = 65536$ midpoint quantifiers (out of 2^{32} type $\langle 1 \rangle$ quantifiers in total), 8 of which are ISOM (out of 64 in total). This shows that in some sense there are *many* midpoints, which seems surprising if you think of them as 'degenerate'. These results are local and have no immediate global versions. Nevertheless, we will see that, in a related sense, there are also *many* global midpoints.

In Keenan (2008) the focus is on type $\langle 1,1 \rangle$ midpoints; as I said, the label comes from certain proportional quantifiers. Following Keenan, Q is *proportional* if the truth value of $Q(A,B)$ depends only on the proportion of Bs among the As (assuming FIN):

(10) For $A, A' \neq \emptyset$, if $|A \cap B|/|A| = |A' \cap B'|/|A'|$ then $Q(A,B) \Leftrightarrow Q(A',B')$.[7]

[6] Keenan's proof uses facts about complete atomic Boolean algebras, but we will see a simpler calculation in section 3 (Corollary 8). Note that in a local approach, the condition corresponding to ISOM is PERM_M; *closure under permutations* of M. One can show that under EXT, ISOM is equivalent to PERM_M holding for all M.

[7] Keenan doesn't mention the requirement $A, A' \neq \emptyset$ (and neither does the definition of proportionality in Keenan and Westerståhl (2011)), but it is needed: if we were to drop it and rewrite the antecedent as $|A \cap B| \cdot |A'| = |A' \cap B'| \cdot |A|$, we would get the consequence that if $Q(\emptyset,\emptyset)$ then $Q(A',B')$ for all A',B', and if not $Q(\emptyset,\emptyset)$ then $Q(A',B')$ holds for no A',B', rendering the notion of proportionality useless.

We note that a (CONSERV and EXT) proportional quantifier is automatically ISOM, since for $A, A' \neq \emptyset$, if $|A \cap B| = |A' \cap B'|$ and $|A - B| = |A' - B'|$ then $|A \cap B|/|A| = |A' \cap B'|/|A'|$, and for $A = \emptyset$ ($A' = \emptyset$), if $|A \cap B| = |A' \cap B'|$ and $|A - B| = |A' - B'|$ then $A' = \emptyset$ ($A = \emptyset$), and thus trivially $Q(A,B) \Leftrightarrow Q(A'B')$.

Let us define the following *basic* proportional quantifiers:[8]

(11) For $0 \leq p \leq q$ (and $q \neq 0$),

a. $(p/q)(A,B) \Leftrightarrow |A \cap B| > p/q \cdot |A|$

b. $[p/q](A,B) \Leftrightarrow |A \cap B| \geq p/q \cdot |A|$

So (p/q) is *more than p/q'ths of the*, and $[p/q]$ is *at least p/q'ths of the*. These are proportional, but many other quantifiers are too; indeed Keenan observes that the class of proportional quantifiers is closed under Boolean operations, including inner negation. For example,

$$\textit{between one-third and two-thirds of the} \;=\; [1/3] \wedge \neg(2/3)$$

is proportional. That it is also a midpoint follows from

Theorem 1 (Keenan's First Midpoint Theorem)**.** *If $p/q + p'/q' = 1$, then the quantifier* between p/q and p'/q' of the *is a midpoint.*

Thinking of 1/2 as the midpoint, the requirement $p/q + p'/q' = 1$ means that p/q and p'/q' have equal distance to the midpoint,[9] which explains the terminology.

The next step is to generalize this further, noting two things. First, an easy calculation shows

(12) a. $[p/q]\neg \;=\; \neg((q-p)/q)$

b. $(p/q)\neg \;=\; \neg[(q-p)/q]$

Second, we have (collecting some of Keenan's results in one theorem):

Theorem 2 (Keenan's Second Midpoint Theorem)**.**

(a) *For any Q, the quantifiers $Q \wedge Q\neg$ and $Q \vee Q\neg$ are midpoints.*

(b) *The class of midpoints is closed under Boolean operations, including inner negation.*

(a) is an immediate corollary of (6-c) and (6-a): $(Q \wedge Q\neg)\neg = Q\neg \wedge Q\neg\neg = Q\neg \wedge Q = Q \wedge Q\neg$, and similarly for $Q \vee Q\neg$. Then we note that if the assumption of Theorem 1 is satisfied we have $p'/q' = (q-p)/q$, and hence, using (12-a), that

$$\textit{between } p/q \textit{ and } p'/q' \textit{ of the} \;=\; [p/q] \wedge \neg((q-p)/q) \;=\; [p/q] \wedge [p/q]\neg\,,$$

so Theorem 1 follows. Theorem 2(b) also follows by applications of (6).

The midpoint theorems are formulated locally, but the theorems and their proofs extend immediately to a global context. So a global approach adds nothing new to *these* results. But

[8] Elsewhere, e.g. in Keenan and Westerståhl (2011), it is required that $0 < p < q$. Allowing $p = 0$ or $p = q$ makes for greater generality, which turns out to be useful; see the discussion of (16) below.

[9] Note that since $0 \leq p/q \leq p'/q' \leq 1$, we have $p/q \leq 1/2$ and $p'/q' \geq 1/2$, and so $1/2 - p/q = p'/q' - 1/2$, since $p/q + p'/q' = 1$.

Keenan also raises the natural question of a useful *characterization* of the property of being a midpoint, and conjectures that the answer has something to do with proportionality. Here is where I think a global perspective helps.

Keenan (2005, 2008) also presents a number of striking examples, such as the following equivalent pairs:

(13) a. More than three out of ten and less than seven out of ten teachers are married.
b. More than three out of ten and less than seven out of ten teachers are not married.

(14) a. Between 40 and 60 per cent of the students passed.
b. Between 40 and 60 per cent of the students didn't pass.

(15) a. Either all or none of the students will pass that exam.
b. Either all or none of the students will not pass that exam.

(16) a. Some but not all of the professors are on leave.
b. Some but not all of the professors are not on leave.

(17) a. Either exactly five or else all but five students came to the party.
b. Either exactly five or else all but five students didn't come to the party.

(18) a. Exactly three of the six students passed the exam.
b. Exactly three of the six students didn't pass the exam.

As Keenan points out, (13)–(16) are proportional instances of Theorem 2(a). For example, we see that *some* = $(0/1)$, so *some but not all* = $(0/1) \wedge (0/1)\neg$. (Here is where allowing $p = 0$ in (11) is useful!) But he also shows that (17) and (18) do *not* involve proportional quantifiers, thus severing the tie between proportionality and midpoints. As we will see, there seems to be no hope of maintaining that tie.

3 Midpoints in the number triangle

The number triangle provides a thoroughly global view of quantifiers, but it presupposes CONSERV, EXT, ISOM, and FIN. Let us see what proportionality and midpoints look like from this perspective. I'm not sure there is a useful visual way to think of proportionality in general, as defined by (10), i.e. the condition that

(19) if $k+m, k'+m' > 0$ and $m/(k+m) = m'/(k'+m')$, then $Q(k,m) \Leftrightarrow Q(k',m')$.

But the basic proportionals $[p/q]$ and (p/q) from (11) are easy to 'see' in the number triangle, for example, *most* = $(1/2)$ was drawn in Fig. 3. And the midpoint property is beautifully represented in the triangle. First, note that the inner negation of Q becomes the *converse* of Q as a relation between numbers:

(20) $Q\neg(k,m) \Leftrightarrow Q(m,k)$

Thus,

(21) Q is a midpoint iff for all k and m, $Q(k,m) \Leftrightarrow Q(m,k)$.

So the midpoint property says something about how Q must behave *on each diagonal*, where the *diagonal at level* n is $(n,0),(n-1,1),\ldots,(1,n-1),(0,n)$. For example, here are some 'midpoint patterns':

$$
\begin{array}{ccccccccc}
+ & - & + & - & + & - & + & - & + \\
\\
+ & - & - & - & - & - & - & - & + \\
\\
+ & + & - & - & - & - & - & + & + \\
\\
- & - & - & - & + & - & - & - & -
\end{array}
$$

Figure 4: Some midpoint patterns (at level 8)

Imagine a vertical line drawn from (0,0) in the number triangle, thus passing through (1,1), (2,2), (3,3), ..., and between (1,0) and (0,1), between (2,1) and (1,2), between (3,2) and (2,3), etc. Let the *left part* of the number triangle consist of all the points to the left of that line, *including* the points on the line itself. (So, for example, (2,2) and (3,2) are in the left part, but (2,3) is not.) Then, essentially by just 'looking' in the number triangle, we have the following result.

Theorem 3. (CONSERV, EXT, ISOM, FIN) *The following are equivalent:*

(a) *Q is a midpoint.*

(b) *For some Q', $Q = Q' \vee Q'\neg$.*

(c) *For some subset Q' of the left part of the number triangle, Q is the union of Q' and its* mirror image*, i.e. $Q'\neg$.*

That (b) implies (a) is the first part of Keenan's Second Midpoint Theorem,[10] and the converse implication is trivial (with $Q' = Q$). And (c) essentially just restates this in a more pictorial way, noting that Q' can always be taken as a subset of the left part. So there is really nothing new in this theorem, except for the visual aid. But that aid, it seems to me, brings some insight.

First, I think we must abandon all hope of connecting midpoints in general to proportionality. *Any* subset of the left part yields a midpoint, regardless of requirements like (19).

Second, we see that also from a global perspective there are *many* midpoints. There are $2^{\aleph_0}$ subsets of the left part of the number triangle. Hence:

Corollary 4. *There are $2^{\aleph_0}$ midpoints, even if only finite universes are considered, and even if* CONSERV, EXT, *and* ISOM *are imposed.*

Third, we can sharpen the First Midpoint Theorem to an equivalence:

Corollary 5. *The quantifier* between p/q and p'/q' of the *is a midpoint iff $p/q + p'/q' = 1$.*

[10] The second part is also easily 'seen' to be true in the triangle. For Q is a midpoint iff it is symmetric as a relation between numbers, and symmetry is obviously preserved by the Boolean operations.

This doesn't really require the number triangle, but using the triangle makes it fairly obvious (I won't give details here) that if $p/q + p'/q' \neq 1$, one can find a counter-example to the midpoint property by looking at the diagonal at level n, for a large enough n.

Fourth, we can see why some common quantifiers *cannot* be midpoints. Keenan (2008) proves that no non-trivial *intersective* quantifier can be a midpoint.

Definitions:

a. Q is *intersective* if $A \cap B = A' \cap B'$ entails $Q(A,B) \Leftrightarrow Q(A',B')$
b. $\mathbf{1}_M(A,B)$ holds for all $A,B \subseteq M$ ($\mathbf{1}_M$ is the *trivially true* quantifier on M)
c. $\mathbf{0}_M(A,B)$ holds for no $A,B \subseteq M$; ($\mathbf{0}_M$ is the *trivially false* quantifier on M)

Corollary 6 (Keenan)**.** *If Q is an intersective midpoint, then on each M, Q_M is either $\mathbf{1}_M$ or $\mathbf{0}_M$.*

The result is easily provable in the number triangle, but in this case, Keenan's proof of the more general fact is just as simple: Note first that if Q is intersective then $Q\neg$ is *co-intersective*, i.e. the truth value of $Q(A,B)$ depends only on $A - B$. Now suppose Q is an intersective midpoint. Then, for any M and any $A,B \subseteq M$, $Q_M(A,B) \Leftrightarrow Q_M(A \cap B, M)$ (since Q is intersective) $\Leftrightarrow Q_M(\emptyset, M)$ (since $Q = Q\neg$ is co-intersective), so Q_M is either $\mathbf{1}_M$ or $\mathbf{0}_M$.

Other similar results are evident by looking in the number triangle; I give one more example. First, a definition:

Q is *right monotone* if $Q(A,B)$ and $B \subseteq B'$ implies $Q(A,B')$.

Most common English Dets denote right monotone quantifiers, or Boolean combinations of such quantifiers (see Peters and Westerståhl (2006), ch. 5, for a fuller statement). However:

Corollary 7. *If Q is a right monotone midpoint, then on each M, Q_M is either $\mathbf{1}_M$ or $\mathbf{0}_M$.*

This fact is obvious in the number triangle, but again there is a very simple proof without any extra conditions on Q or on the size of universes: Suppose $Q_M(A,B)$ holds. By right monotonicity, $Q_M(A,M)$. Since $Q = Q\neg$, we get $Q_M(A,\emptyset)$. Thus, by right monotonicity again, $Q_M(A,C)$ holds for any $C \subseteq M$.[11]

Finally, let us get back to counting quantifiers on a given universe. I said that the number triangle embodies a global perspective, but it can be used locally too. For CONSERV, EXT, and ISOM type $\langle 1,1 \rangle$ quantifiers on an n-element universe M, just look at the finite triangle up to and including the n'th diagonal. There are $(n+1)(n+2)/2$ pairs in this triangle, so the total number of such quantifiers on M is

$$2^{(n+1)(n+2)/2}$$

And a simple calculation shows that if n is odd, the number of pairs in the left part of the triangle is $(n+1)(n+3)/4$, whereas if n is even it is $n(n+4)/4+1$.[12]

[11] It often happens that a result obtained by looking at the number triangle turns out to hold under more general conditions. (That's how I came to Corollary 7.) The above results answer natural questions, but in other cases one would hardly ever have thought of the more general result, had it not been suggested by the number triangle; see Peters and Westerståhl (2006), ch. 5, for a few examples.

[12] These calculations use essentially nothing more than the fact that $1+2+3+\ldots+k = k(k+1)/2$.

We can also do this for ISOM type $\langle 1\rangle$ quantifiers on M. Then only the diagonal at level $|M|$ is relevant.[13] It has $n+1$ pairs, of which $(n+1)/2$ belong to the left part if n is odd, and $(n+2)/2$ belong to the left part if n is even. Thus:

Corollary 8. *Let M be a universe with n elements. If n is odd, there are $2^{(n+1)(n+3)/4}$ CONSERV, EXT, and ISOM type $\langle 1,1\rangle$ midpoint quantifiers on M, and $2^{(n+1)/2}$ ISOM type $\langle 1\rangle$ midpoint quantifiers on M. If n is even, the corresponding numbers are $2^{n(n+4)/4+1}$ and $2^{(n+2)/2}$, respectively.*

4 Self-duality

Let me spell out definition (7-b) in some more generality:

(22) a. A type $\langle 1,1\rangle$ Q is self-dual iff $\forall M\,\forall A,B\subseteq M\,(Q_M(A,B)\Leftrightarrow\neg Q_M(A,M-B))$.
b. A type $\langle 1\rangle$ Q is self-dual iff $\forall M\,\forall B\subseteq M\,(Q_M(B)\Leftrightarrow\neg Q_M(M-B))$.

The problem with self-dual quantifiers is that they almost never exist.

Theorem 9.

(a) *No* CONSERV *type $\langle 1,1\rangle$ quantifier is self-dual.*

(b) *No* ISOM *type $\langle 1,1\rangle$ or type $\langle 1\rangle$ quantifier is self-dual.*

(c) *Montagovian individuals, i.e. type $\langle 1\rangle$ quantifiers of the form $(I_a)_M(B)\Leftrightarrow a\in B$, are not self-dual.*

(d) *Type $\langle 1\rangle$ quantifiers interpreting quantified DPs, i.e. of the form Q^A for some* CONSERV *and* EXT *type $\langle 1,1\rangle$ Q, are not self-dual.*

(a): If Q is CONSERV, then (22-a) requires

$$Q_M(A,B)\Leftrightarrow\neg Q_M(A,A-B)$$

to hold, which is impossible for $A=B=\emptyset$. (b): If Q is ISOM, choosing A,B,M such that $|A-B|=|A\cap B|=|B-A|=|M-(A\cup B)|$ will yield a counter-example to (22-a), and similarly in the type $\langle 1\rangle$ case. (c): As to I_a, choose M such that $a\notin M$: then, for any $B\subseteq M$, $(I_a)_M(B)$ and $(I_a)_M(M-B)$ are both false, contrary to what (22-b) requires. (d): Finally, the quantifiers Q^A are defined by

$$Q^A_M(B)\Leftrightarrow Q_{A\cup M}(A,B)\ ^{14}$$

Choose A disjoint from M. Then, using the conservativity of Q, one easily sees that $Q^A_M=Q^A_M\neg$ (so Q^A_M is a midpoint), contradicting self-duality.

As a bonus, we obtain from Theorem 9(a) a final characterization of midpoints.

[13] Now the binary relation corresponding to Q is

(i) $Q(k,m)\Leftrightarrow$ for some M and some $B\subseteq M$ with $|M-B|=k$ and $|B|=m$, $Q_M(B)$

[14] See Peters and Westerståhl (2006), ch. 4.5.5, for arguments why this is the correct definition, rather than, say, $(Q^A)_M(B)\Leftrightarrow A\subseteq M\ \&\ Q_M(A,B)$. However, also with the latter definition, Q^A cannot be self-dual.

Corollary 10. (CONSERV) *Q is a midpoint iff square(Q) has 2 elements.*

However, Theorem 9 may seem very surprising, in view of the fact that self-dual quantifiers are often discussed in the linguistic literature. For example, Barwise and Cooper (1981) point out that since self-duality means that $\neg Q = Q\neg$, we have an immediate semantic explanation of why negation always has wide scope over self-dual quantifiers, such as I_a. But there is no contradiction here, since Barwise and Cooper are talking about *local* quantifiers, and if $a \in M$, then $(I_a)_M$ is indeed self-dual, in the sense that *on such an M*,

$$\forall B \subseteq M\,((I_a)_M(B) \Leftrightarrow \neg(I_a)_M(M-B))$$

Keenan (2005) also discusses local self-dual type $\langle 1 \rangle$ quantifiers, noting, however, that the ISOM ones rarely exist (he establishes a local version of Theorem 9(b)). Only when $|M| = n$ is odd can you get some self-dual ISOM local quantifiers, like *at least* $(n+1)/2$ *things*, or *at least* $n-1$ *or between 2 and* $(n+1)/2$ *things* (as is seen by looking at the diagonal at level n in the number triangle, and recalling that the condition to satisfy is $(n-k,k) \in Q \Leftrightarrow (k,n-k) \notin Q$).

But at least for proper names interpreted as Montagovian individuals, local self-duality seems like a significant property, which goes to show that sometimes a local view of quantifiers can be rewarding even when there is no reasonable global alternative.

Conclusion

Midpoint quantifiers, discovered (though not named in this way) by Keenan, are a curious and interesting phenomenon, on the borderline between linguistics and logic. I do believe that a global perspective, with the help of the number triangle, offers insights into their properties and distribution. But perhaps this is partly a matter of taste; Keenan is probably so used to working with Boolean algebras that he thinks that framework is easier to visualize. In any case, I have claimed here that for at least one question, concerning a possible connection between midpoints and proportionality, the global view is preferable and in fact suggests a (negative) answer. But I also noted the contrast with respect to a seemingly very similar issue (similar from the point of view of the square of opposition), that of self-dual quantifiers, where the local perspective is essentially the only one in which they even exist. My aim has not been to say that one perspective is preferable to the other, but rather to note the difference between them, and that both have their uses, with certain facts about quantification and negation as paradigmatic examples.

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