From Constants to Consequence, and Back

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Abstract

Bolzano's definition of consequence in effect associates with each set X of symbols (in a given interpreted language) a consequence relation $\Rightarrow_X$. We present this in a precise and abstract form, in particular studying *minimal* sets of symbols generating $\Rightarrow_X$. Then we present a method for going in the other direction: *extracting* from an arbitrary consequence relation $\Rightarrow$ its associated set $C_{\Rightarrow}$ of constants. We show that this returns the expected logical constants from familiar consequence relations, and that, restricting attention to sets of symbols satisfying a strong minimality condition, there is an isomorphism between the set of strongly minimal sets of symbols and the set of corresponding consequence relations (both ordered under inclusion).

1 Introduction

In virtually all accounts of logical consequence, the notion of a logical constant is conceptually prior. Whether consequence is defined model-theoretically (as truth preservation under reinterpretation of the non-constants), or proof-theoretically (in terms of a system of axioms and rules), a selection of logical symbols or operations is made at the start.[1] This leaves the familiar problem of *how* this selection is made: What is it about a meaningful symbol or word that makes it earn the label 'logical'?[2]

This problem is particularly visible in the model-theoretic account, and even more so in Bolzano's original substitutional version. In contrast with standard modern presentations, Bolzano emphasized that *any* set X of symbols can be

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[1] The referee pointed out that some algebraic approaches, where the consequence relation is a partial order and logical constants are definable in terms of it (e.g. as suprema, infima, and complement), may be exceptions to this rule.

[2] See MacFarlane (2009) for a recent survey of approaches to the issue of logicality.

selected as constants, resulting in a corresponding consequence relation $\Rightarrow_X$. The issue of how to choose X then becomes acute.

Instead of seeing this choice as a problem we may view consequence as a *ternary* relation: between the (set of) premises, the conclusion, and X.[3] An equivalent perspective, to be adopted here, is to see $\Rightarrow_{\_}$ as a *function* from sets of symbols to (binary) consequence relations.

My focus in the present paper, however, is also on a function going in the opposite direction: take a consequence relation as *given*, and see if we can *extract*, in a systematic way, the symbols it treats as constants. This perspective, although hardly explored in the literature, is rather natural: you could argue that taking a consequence relation as primitive is more intuitive than starting from a theoretical concept of logicality. In fact, I believe the two perspectives complement each other, and that the study of their interplay may provide insights both in the notion of logicality and in the abstract study of consequence relations.

Since the territory is largely unexplored, it is not immediate what the right definitions are, or what results to expect. Two guiding ideas, however, are the following: (1) The two perspectives, from constants to consequence, and back, should be *inverse* to each other in some suitable sense; (2) The method of extracting constants should yield the usual sets of constants when applied to familiar consequence relations, such as classical or intuitionistic first-order consequence. We will see (Section 4) that (2) indeed holds for the method chosen here, and that (1) holds to a limited extent.

But how is it possible to extract the constants from a given consequence relation $\Rightarrow$? The basic idea is this: *Constants are symbols such that replacing them by other symbols (or reinterpreting them) has the potential to destroy the validity of $\Rightarrow$-inferences.* Consider:

(1) Most logicians are mathematicians.
Most logicians are familiar with Lindström's Theorem.
Hence: Some mathematicians are familiar with Lindström's Theorem.

If you think this is valid (as you should), you presumably also think inferences obtained from (1) by replacing expressions like 'logicians', 'mathematicians', etc. by other expressions of the same category are also valid, for instance,

(2) Most Stanford students are tennis players.
Most Stanford students own white sports cars.
Hence: Some tennis players own white sports cars.

One the other hand, replacing words like 'most' or 'some' clearly need not preserve validity; for example, no one would think the following is valid:

(3) No logicians are mathematicians.
No logicians are familiar with Lindström's Theorem.

[3] This perspective on Bolzano's notion of consequence is explored in van Benthem (2003).

Hence: Some mathematicians are familiar with Lindström's Theorem.

However, even though it is hardly disputable which words are constants and which are not in a particular inference like (1) or (2), it is less clear how this could be turned into a general definition. In Peters and Westerståhl (2006), ch. 9, it was suggested that a symbol is constant if *every* $\Rightarrow$-inference where it occurs essentially can be destroyed by replacing it. As noted there, the qualification about essential occurrences is crucial. This can be illustrated by the following examples from predicate logic:

(4) a. $Pa \;\Rightarrow\; Pa \vee \exists x Rx$

b. $\exists x Px, \forall x(Px \leftrightarrow Rx) \;\Rightarrow\; \exists x Rx$

Both of these hold when $\Rightarrow$ is classical first-order consequence, but in each of them, $\exists$ can be replaced by an arbitrary type $\langle 1 \rangle$ quantifier Q without destroying the inference. Cases like (4a) are fairly easy to set aside: (4a) is an instance of a more general inference where $\exists$ doesn't occur ($\varphi \Rightarrow \varphi \vee \psi$). But (4b), which is a kind of *extensionality axiom*, seems much harder to eliminate on syntactic grounds.

In this paper, we avoid this difficulty by weakening the requirement: it is enough that *some* $\Rightarrow$-inference can be destroyed by suitably replacing the symbol in question. Even so, there turn out to be different ways in which this idea can be implemented. What I will do here is to present the perhaps simplest such implementation, under some fairly restrictive assumptions. This will give the flavor of the approach, and also indicate at which points it could be improved.

2 Background assumptions

2.1 Language

We assume that a fixed *interpreted language* is given, which has a set *Sent* of *sentences*, in which (atomic) *symbols* from a countable set *Symb* occur. For definiteness, think of sentences a finite strings of symbols and possibly other signs. Let $u, v, u', \ldots$ vary over *Symb*, $\varphi, \psi, \ldots$ over *Sent*, and $\Gamma, \Delta, \ldots$ over sets of sentences, and let

$$V_\varphi$$

be the set of symbols occurring in φ; similarly $V_\Gamma = \cup\{V_\varphi : \varphi \in \Gamma\}$.

Very little needs to be assumed about how sentences are structured; the important thing is that there is a notion of *replacement* of symbols by symbols.[4] Such replacements should be 'appropriate', and the easiest way to think of this is that it should *respect categories*. Thus, assuming that *Symb* is partitioned

[4]This is a simplifying assumption; a natural generalization is to consider replacing primitive symbols by complex expressions of the same category.

into a set of categories, a *replacement* is a partial function ρ from *Symb* to *Symb* such that for $u \in dom(\rho)$, u and $\rho(u)$ belong to the same category. The result of applying ρ to φ, written

$$\varphi[\rho]$$

is simply the result of replacing *each occurrence of* u in φ by $\rho(u)$. Here it is convenient to always assume that $V_\varphi \subseteq dom(\rho)$ — in words, ρ is a replacement *for* φ — so that ρ is the identity on symbols that don't get replaced. We may then assume that the following conditions hold:[5]

(5)
a. If ρ is a replacement for φ, $\varphi[\rho] \in Sent$ and $V_{\varphi[\rho]} = range(\rho \restriction V_\varphi)$
b. $\varphi[id_{V_\varphi}] = \varphi$
c. If ρ, σ agree on V_φ, then $\varphi[\rho] = \varphi[\sigma]$.
d. $\varphi[\rho][\sigma] = \varphi[\sigma\rho]$, when σ is a replacement for $\varphi[\rho]$

Finally, that the language is interpreted is taken to entail that each sentence is either *true* or *false*.

2.2 Consequence relations

As another simplifying assumption, we take consequence relations to hold between finite sets of sentences and sentences (rather than, say, allowing infinite sets of premises, or considering sequences instead of sets).

Definition 1
A relation $R \subseteq P^{<\omega}(Sent) \times Sent$ is
(i) *reflexive* iff for all $\varphi \in Sent$, $\varphi\, R\, \varphi$;[6]
(ii) *transitive* iff whenever $\Delta\, R\, \varphi$ and $\Gamma\, R\, \psi$ for all $\psi \in \Delta$, we have $\Gamma\, R\, \varphi$;
(iii) *monotone* iff $\Delta\, R\, \varphi$ and $\Delta \subseteq \Gamma$ implies $\Gamma\, R\, \varphi$;
(iv) *truth-preserving* iff whenever $\Gamma\, R\, \varphi$ and (every sentence in) Γ is true, φ is also true.

Definition 2
A *consequence relation* is a reflexive, transitive, monotone, and truth-preserving relation (between finite sets of sentences and sentences). We let $\Rightarrow, \Rightarrow', \ldots$ vary over the set *CONS* of consequence relations.

Define:

(6)
a. $\Gamma \Rightarrow^{max} \varphi$ iff it is not the case that Γ is true and φ is false.
b. $\Gamma \Rightarrow^{min} \varphi$ iff $\varphi \in \Gamma$.

[5] These are essentially the conditions in Peter Aczel's notion of a *replacement system* from Aczel (1990).
[6] Writing $\psi\, R\, \varphi$ instead of $\{\psi\}\, R\, \varphi$.

$\Rightarrow^{max}$ is essentially *material implication*. One easily verifies the following:

Fact 3
$\Rightarrow^{max}, \Rightarrow^{min} \in CONS$, and $(CONS, \subseteq)$ is partial order with $\Rightarrow^{min}$ as its smallest and $\Rightarrow^{max}$ as its largest element.

If $R_0 \subseteq P^{<\omega}(Sent) \times Sent$, define

(7) $\quad cl(R_0) = \bigcap\{R\colon R \text{ is a consequence relation and } R_0 \subseteq R\}$

Fact 4
If R_0 preserves truth, $cl(R_0)$ is the smallest consequence relation that includes R_0.

Proof. The set $\{R\colon R$ is a consequence relation and $R_0 \subseteq R\}$ is non-empty since it contains $\Rightarrow^{max}$. One readily checks that $cl(R_0)$ is also reflexive, transitive, and monotone. □

3 From constants to consequence (Bolzano style)

3.1 The function $\Rightarrow_{\_}$

The following definition should be familiar to every logician, except possibly for the fact that (a) it is substitutional rather than model-theoretic; (b) it allows any set of symbols to be treated as logical.

Definition 5
For any $X \subseteq Symb$, define the relation $\Rightarrow_X$ by

> $\Gamma \Rightarrow_X \varphi$ iff for every replacement ρ (for Γ and φ) which is the identity on X, if $\Gamma[\rho]$ is true, so is $\varphi[\rho]$.

A relation of the form $\Rightarrow_X$ is called a *Bolzano consequence* (relation); *BCONS* is the set of Bolzano consequences.

It is straightforward to verify the following claims.

Fact 6
(a) *$BCONS \subseteq CONS$*

(b) *In addition, Bolzano consequence is* base monotone, *in that*

$$X \subseteq Y \text{ implies } \Rightarrow_X \subseteq \Rightarrow_Y$$

(c) *$(BCONS, \subseteq)$ is a partial order which has $\Rightarrow_\emptyset$ as its smallest and $\Rightarrow_{Symb}$ as its largest element.*

So $(BCONS, \subseteq)$ is a sub-order of $(CONS, \subseteq)$, and we see that

(8) $\Rightarrow^{max} = \Rightarrow_{Symb}$

It often happens, however, that

$$\Rightarrow^{min} \subsetneq \Rightarrow_{\emptyset}$$

BCONS is usually a proper subset of *CONS*.[7]

The following lemma is trivial but fundamental:

Lemma 7
(Replacement Lemma) *If* $\Gamma \Rightarrow_X \varphi$ *and* ρ *doesn't move any (is the identity on) symbols in* X*, then* $\Gamma[\rho] \Rightarrow_X \varphi[\rho]$.

Proof. Use the composition property (5d) of replacement, noting that if both ρ and σ only move symbols outside X, so does $\sigma\rho$. □

Furthermore, from base monotonicity and (5c) we see that only symbols occurring in premises and conclusion matter for Bolzano consequence:

Lemma 8
(Occurrence Lemma) $\Gamma \Rightarrow_X \varphi$ *if and only if* $\Gamma \Rightarrow_{X \cap V_{\Gamma \cup \{\varphi\}}} \varphi$.

3.2 Examples

3.2.1 Propositional logic

Suppose our language is a standard language of *propositional logic*, whose symbols consist of a suitable set X_0 of connectives (say, $X_0 = \{\neg, \wedge, \vee\}$) and an infinite supply $p_0, p_1, \ldots$ of propositional letters, and let $\models_{PL}$ be the corresponding (classical) consequence relation. The usual definition of $\models_{PL}$ is model-theoretic, but we can 'simulate' it also in the present substitutional setting, where $p_0, p_1, \ldots$ are sentences with fixed truth values. Replacing proposition letters by others amounts to 'assigning' arbitrary truth values to them, under the simple assumption that the sequence of truth values of $p_0, p_1, \ldots$ is *not eventually constant*. The following is then easily verified.

Fact 9
$\Gamma \models_{PL} \varphi$ *iff* $\Gamma \Rightarrow_{X_0} \varphi$

3.2.2 First-order logic

Now suppose the language is that of *first-order logic*, whose symbols consist of a suitable set X_1 of logical constants (say, $X_1 = X_0 \cup \{\exists, \forall, =\}$) and a supply of predicate symbols and individual constants, all with a given interpretation, and let $\models_{FO}$ be the standard (classical) consequence relation.

[7] For example, in propositional logic, $***p \Rightarrow_{\emptyset} *p$, where p is a propositional letter and $*$ any unary truth function, but $***p \not\Rightarrow^{min} *p$.

At first blush, one might think $\Rightarrow_{X_1}$ simply amounts to the relation $\models_{FO_{\text{subst}}}$, i.e. the consequence relation you get with a *substitutional interpretation of the quantifiers*, but this is not quite so. The reason is that in standard definitions of logical consequence with substitutional quantification, as in Dunn and Belnap Jr. (1968), only the quantifiers are interpreted substitutionally, not the rest of the language. Without going into details here, in the substitutional account consequence is defined relative to arbitrary assignments of truth values to atomic sentences, but there is no guarantee that every such assignment can be 'simulated' by replacing predicate symbols and individual constants in our given interpreted language. Only under such a guarantee will $\Gamma \Rightarrow_{X_1} \varphi$ imply $\Gamma \models_{FO_{\text{subst}}} \varphi$ (the converse implication always holds), but this is a very strong assumption on the language, in contrast with the assumption needed to 'simulate' $\models_{PL}$.

Moreover, $\models_{FO_{\text{subst}}}$ is different from $\models_{FO}$: Dunn and Belnap point out that if there are infinitely many individual constants, *FO-validity* (consequence of the empty set) coincides with FO_{subst}-validity, but *FO-consequence* still differs.

In brief, $\models_{FO}$ is a consequence relation, but not a Bolzano consequence.

3.2.3 Two toy examples

The following examples will be used to illustrate some claims later on.

Ex. 1 $\quad Symb = \{a, b\}$
$Sent = \{Raa, Rab, Rba, Rbb\}$
True: Rab, Raa, Rbb
False: Rba

Each of the four subsets of $\{a, b\}$ generates a Bolzano consequence. For example,

$$\emptyset \Rightarrow_{\emptyset} Raa$$

$$\emptyset \Rightarrow_{\{a\}} Rab$$

I will not go through the details here, but it is fairly easy to see that these consequence relations can be exhaustively described as follows:

(9) a. $\Rightarrow_{\emptyset} = cl(\{(\emptyset, Raa), (\emptyset, Rbb)\})$

b. $\Rightarrow_{\{a\}} = \Rightarrow_{\{b\}} = \Rightarrow_{\{a,b\}} = cl(\{(\emptyset, Raa), (\emptyset, Rbb), (\emptyset, Rab)\})$

Ex. 2 $\quad Symb = \{a, b, c\}$
$Sent = \{Raa, Rab, Rac, Rba, Rbb, Rcb, Rca, Rcb, Rcc\}$
True: all except Rba

Again it is (tedious but) fairly straightforward to give short complete descriptions of the corresponding Bolzano consequences. We shall only need the following facts (proofs omitted here):

(10) a. $\Rightarrow_{\emptyset} = cl(\{(\emptyset, Raa), (\emptyset, Rbb), (\emptyset, Rcc)\})$

b. $\Rightarrow_{\{a\}} = cl(\{(\emptyset, Raa), (\emptyset, Rbb), (\emptyset, Rcc), (\emptyset, Rab), (\emptyset, Rac), (\{Rca\}, Rcb), (\{Rba\}, Rbc)\})$

c. $\Rightarrow_{\{a,c\}} = cl(\{(\emptyset, Raa), (\emptyset, Rbb), (\emptyset, Rcc), (\emptyset, Rab), (\emptyset, Rac), (\emptyset, Rbc), (\emptyset, Rca), (\emptyset, Rcb)\})$

3.3 Minimal sets of symbols

Example 1 above illustrates that different sets may generate the same Bolzano consequence. One would expect that sets that are *minimal* in this respect are particularly well behaved. Another natural notion of minimality is the following.

Definition 10
X is *minimal* iff for all $u \in X$, $\Rightarrow_X \neq \Rightarrow_{X-\{u\}}$.

Since $\Rightarrow_{X-\{u\}} \subseteq \Rightarrow_X$, minimality of X in this sense means that if any one of its symbols is left out, a smaller consequence relation results. We first observe that this is in fact the same notion as the one first mentioned.

Fact 11
X is minimal iff no proper subset of X generates the same consequence relation.

Proof. Suppose the condition on the right-hand side holds, and take $u \in X$. Since $X - \{u\}$ is a proper subset of X, we must have $\Rightarrow_{X-\{u\}} \neq \Rightarrow_X$, so X is minimal.

Conversely, suppose X is minimal, $\Rightarrow_Y = \Rightarrow_X$, and $Y \subseteq X$. We must show that $Y = X$. If $u \in X - Y$ then $Y \subseteq X - \{u\}$, and so we have (by base monotonicity)

$$\Rightarrow_X \; = \; \Rightarrow_Y \; \subseteq \; \Rightarrow_{X-\{u\}} \; \subseteq \; \Rightarrow_X$$

which means that $\Rightarrow_{X-\{u\}} = \Rightarrow_X$, contradicting the minimality of X. Thus, $Y = X$. □

In Examples 1 and 2 of Section 3.2.3, all the singleton sets are minimal, since in each case, $\Rightarrow_{\{x\}}$ is distinct from $\Rightarrow_{\emptyset}$. Note that the empty set is always trivially minimal. The set $\{a, b\}$ in Example 1 is not minimal, but it has a minimal subset generating the same consequence relation (in fact, it has two such subsets; cf. (9)). We now see that such a subset always exists.

Proposition 12
Every $X \subseteq Symb$ has a subset which is minimal among those generating $\Rightarrow_X$.

Proof. Suppose $X = \{u_1, u_2, \ldots\}$. Define sets X_n for each n by

$$\begin{cases} X_0 & = X \\ X_{n+1} & = \begin{cases} X_n & \text{if } \Rightarrow_{X_n} \not\subseteq \Rightarrow_{X_n - \{u_n\}} \\ X_n - \{u_n\} & \text{otherwise} \end{cases} \end{cases}$$

Thus, $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$. By a simple induction,

(11) For all n, $\Rightarrow_{X_n} = \Rightarrow_X$.

Let $X^* = \cap_{n\in\omega} X_n$.

(12) $\Rightarrow_{X^*} = \Rightarrow_X$

This is clear from (11) if X is finite; then $X^* = X_k$ for some k. In the general case, argue as follows. Since $X^* \subseteq X$, it suffices to show $\Rightarrow_X \subseteq \Rightarrow_{X^*}$. Suppose that $\Gamma \Rightarrow_X \varphi$, that ρ only moves symbols outside X^*, and $\Gamma[\rho]$ is true. We must show that $\varphi[\rho]$ is true. Since Γ is finite, and only finitely many symbols occur in a sentence, and we only need to care about symbols occurring Γ and φ (Occurrence Lemma), we can assume

$$\rho = \rho' \cup \{(u_{i_1}, v_1), \ldots, (u_{i_k}, v_k)\}$$

where $u_{i_1}, \ldots, u_{i_k} \notin X^*$ and ρ' only moves symbols outside X. Take n such that $u_{i_1}, \ldots, u_{i_k} \notin X_n$. We have $\Gamma \Rightarrow_{X_n} \varphi$ by (11), and ρ only moves symbols outside X_n, so $\Gamma[\rho] \Rightarrow_{X_n} \varphi[\rho]$ (Replacement Lemma). Thus, $\varphi[\rho]$ is true, and (12) is proved.

Finally,

(13) X^* is minimal.

Take $u_i \in X^*$. Then $u_i \in X_{i+1}$, so $\Rightarrow_{X_i} \not\subseteq \Rightarrow_{X_i - \{u_i\}}$. By (11) and (12),

$$\Rightarrow_{X^*} \not\subseteq \Rightarrow_{X_i - \{u_i\}}$$

But $X^* - \{u_i\} \subseteq X_i - \{u_i\}$, so $\Rightarrow_{X^* - \{u_i\}} \subseteq \Rightarrow_{X_i - \{u_i\}}$. It follows that $\Rightarrow_{X^*} \not\subseteq \Rightarrow_{X^* - \{u_i\}}$. □

Thus, if we restrict attention to minimal subsets of *Symb*, no consequence relation of the form $\Rightarrow_X$ will be left out.

3.4 Strongly minimal sets

A stricter notion of minimality also proves to be useful:

Definition 13
X is *strongly minimal* iff for all $u \in X$ there is Γ, φ, and u' such that $\Gamma \Rightarrow_X \varphi$, $\Gamma[u/u']$ is true, but $\varphi[u/u']$ is false.[8]

Note that $\emptyset$ is always strongly minimal.

Fact 14
If X is strongly minimal, it is minimal.

[8] u/u' is the replacement which maps u to u' but is the identity on all other symbols.

Proof. Take $u \in X$ and let Γ, φ, and u' be as above. Then the replacement $\rho = \{(u, u')\}$ shows that $\Gamma \not\Rightarrow_{X-\{u\}} \varphi$. □

Strong minimality says that $\Rightarrow_X \subseteq \Rightarrow_{X-\{u\}}$ fails in a particular way: a counter-example exists which involves replacing only u.

In Example 1 (Section 3.2.3), the two minimal sets $\{a\}$ and $\{b\}$ are not strongly minimal. For example, in the case of $\{a\}$, it follows from (9b) that the only inferences we have to worry about are $\emptyset \Rightarrow_{\{a\}} Raa$ and $\emptyset \Rightarrow_{\{a\}} Rab$, but replacing a gives the true conclusion *Rbb*. Similarly for $\{b\}$. The set $\{a, b\}$ is not even minimal. So $\{a, b\}$ has no strongly minimal subset other than $\emptyset$. Moreover, $\Rightarrow_{\{a,b\}}$ is not of the form $\Rightarrow_X$ for any strongly minimal X.

Thus, not all Bolzano consequences are of the form $\Rightarrow_X$ for strongly minimal X. However, those of this form are particularly well behaved:

Lemma 15
If X is strongly minimal then, for all Y,

$$X \subseteq Y \text{ iff } \Rightarrow_X \subseteq \Rightarrow_Y$$

Proof. It always holds that $X \subseteq Y$ implies $\Rightarrow_X \subseteq \Rightarrow_Y$. So suppose X is strongly minimal and $\Rightarrow_X \subseteq \Rightarrow_Y$. Take $u \in X$, and let Γ, φ, and u' be as in the definition of strong minimality. So $\Gamma \Rightarrow_X \varphi$, and hence $\Gamma \Rightarrow_Y \varphi$. But if $u \notin Y$, it would follow that $\Gamma[u/u'] \Rightarrow_Y \varphi[u/u']$, by the Replacement Lemma. That is impossible, since $\Gamma[u/u']$ is true and $\varphi[u/u']$ is false. Thus, $u \in Y$. □

We immediately get:

Corollary 16
The mapping $\Rightarrow_{\_}$ is one-one on strongly minimal sets.

4 From consequence to constants

We now turn to the problem of extracting the constants of a given consequence relation. As indicated in the Introduction, the method of extraction chosen in this paper is the following:

Definition 17
For $\Rightarrow \in$ *CONS*, let $C_{\Rightarrow}$ be the set of symbols u such that there are Γ, φ, and u' such that $\Gamma \Rightarrow \varphi$ but $\Gamma[u/u'] \not\Rightarrow \varphi[u/u']$.

So a constant in this sense is such that some way of replacing it, but leaving all other symbols as they are, destroys some valid inference.

4.1 Some general facts

We first note that all constants in this sense relative to a Bolzano consequence relation $\Rightarrow_X$ belong to X.

Lemma 18
For all $X \subseteq Symb$, $C_{\Rightarrow_X} \subseteq X$.

Proof. If $u \in C_{\Rightarrow_X}$, there are Γ, φ, and u' such that $\Gamma \Rightarrow_X \varphi$ but $\Gamma[u/u'] \not\Rightarrow_X \varphi[u/u']$. But if $u \notin X$ we would have $\Gamma[u/u'] \Rightarrow_X \varphi[u/u']$ by the Replacement Lemma. So $u \in X$. □

In particular, we always have

(14) $\quad C_{\Rightarrow_\emptyset} = \emptyset$

In Example 1, $C_{\Rightarrow_X} = \emptyset$ for all $X \subseteq Symb$, since the only way to destroy an inference in this example is to replace *two* symbols (by each other). This could be taken to indicate that something is amiss here, either with the example or with our definitions of $\Rightarrow_X$ and $C_\Rightarrow$ (or both). A further hint to this effect is provided by the following observation: Add a *copy* of b in the example, i.e. a new symbol c which behaves with respect to well-formedness and truth exactly as b.[9] This gives us the language of Example 2, but with the difference that both *Rba* and *Rca* are false, whereas all the other sentences are true. Now we can see that

(15) $\quad C_{\Rightarrow_{\{a\}}} = \{a\}$

For the fact that $\emptyset \Rightarrow_{\{a\}} Rac$ (replacing c cannot produce a false conclusion) but $\emptyset \not\Rightarrow_{\{a\}} Rbc$ (replace c by a) in this case shows that $a \in C_{\Rightarrow_{\{a\}}}$, and then (15) follows by Lemma 18. Similarly, one can show that

(16) $\quad C_{\Rightarrow_{\{b\}}} = \{b\}$

(17) $\quad C_{\Rightarrow_{\{a,b\}}} = \{a, b\}$

In other words, by adding a copy of b to the language, which seems like a rather harmless thing to do, Definitions 5 and 17 do produce the 'right' constants.

We shall not explore this further here, but instead note that under the assumption of strong minimality, we get a strengthening of Lemma 18:

Fact 19
If X is strongly minimal, then $X = C_{\Rightarrow_X}$.

Proof. By Lemma 18, $C_{\Rightarrow_X} \subseteq X$. In the other direction, take $u \in X$, and suppose Γ, φ, and u' are as in the definition of strong minimality: $\Gamma \Rightarrow_X \varphi$, $\Gamma[u/u']$ is true, but $\varphi[u/u']$ is false. But then trivially $\Gamma[u/u'] \not\Rightarrow_X \varphi[u/u']$, since $\Rightarrow_X$ preserves truth. Thus, $u \in C_{\Rightarrow_X}$. □

We can use Example 2 in Section 3.2.3 to see that the converse of Fact 19 fails. First, the argument above for (15) shows that this holds in Example 2 as well.[10] However,

[9] Intuitively, c is a new name of the individual named by b.
[10] In fact, each of (15)–(17) holds for Example 2.

(18) $\{a\}$ is not strongly minimal.

To see this, we must look at all non-trivial $\Rightarrow_{\{a\}}$-inferences involving a, and check that just replacing a by b or by c does not yield true premises and a false conclusion. But this follows by inspecting the description (10b) of those inferences in Section 3.2.3. Thus, there are sets X satisfying $X = C_{\Rightarrow_X}$ which are not strongly minimal.

4.2 Application to some familiar consequence relations

As stated in the Introduction, we should check that Definition 17 is correct at least in the sense of returning the expected logical constants from some familiar consequence relations. But we must also watch out for undesired effects, in Bolzano's substitutional framework, of limited expressive resources of the language. There is one case in particular, where a definition of this kind obviously will not work:

Fact 20
If u is unique in its category, i.e. if there are no other symbols of the same category in the language, then, for any consequence relation $\Rightarrow$, $u \notin C_{\Rightarrow}$.

A typical case is *negation*: most logical languages have no other symbols of the category 'unary truth function'. We will see, however, that assuming there is just one more such symbol will remove this problem.

4.2.1 Classical propositional logic

Consider the classical consequence relation $\models_{PL}$ from Section 3.2.1. In view of Fact 20, however, let us assume there is also a symbol for the *falsum* unary truth function $\mathrm{F}_{\langle t,t\rangle}$,[11] so that $X_0 = \{\neg, F_{\langle t,t\rangle}, \wedge, \vee\}$. This does not affect Fact 9, i.e. that, assuming the sequence of truth values of sentential symbols is not eventually constant, $\models_{PL}$ coincides with $\Rightarrow_{X_0}$.

So the result we want is $C_{\Rightarrow_{X_0}} = X_0$, i.e. we must show that $X_0 \subseteq C_{\Rightarrow_{X_0}}$. But it is obvious that for each symbol in X_0 we can destroy some valid inference by replacing it. For example, replacing $\wedge$ by $\vee$ destroys the validity of $p \wedge q \models_{PL} p$. We can also see that X_0 is strongly minimal (cf. Fact 19). In fact, much more can be shown. Let a *propositional language* be such that besides propositional letters $p_0, p_1, \ldots$, it contains symbols for some unary and binary truth functions, and that sentences are built in the usual way and assigned truth values according to the usual truth tables.

Proposition 21
In a propositional language such that $X_0 \subseteq Symb$, every subset of $Symb$ is strongly minimal.

[11] I.e. the constant function mapping each truth value to F.

The proof of this is not difficult, though somewhat lengthy, and will be omitted here.

To appreciate what Proposition 21 says, note first that it also applies to cases when some propositional letters are treated as constants. It is of course familiar to use the symbols T and/or F in that way; what the result says is that whatever propositional letters are included in X, the Bolzano consequence relation $\Rightarrow_X$ is such that its constants, according to Definition 17, are precisely the elements of X.

Note further that many of the consequence relations $\Rightarrow_X$ covered by Proposition 21 will be quite unnatural. For example, if $\rightarrow \in Symb$ (denoting the binary material implication truth function) but $\rightarrow \notin X$, then

$$\{\varphi, \varphi \rightarrow \psi\} \not\Rightarrow_X \psi,$$

since $\Rightarrow_X$ treats $\rightarrow$ as variable, i.e. replaceable by any binary truth function symbol. In general, the 'reasonable' cases will be those relations $\Rightarrow_X$ where $X = Symb - \{p_0, p_1, \ldots\}$. That is, in those cases, $\Rightarrow_X$ coincides with classical propositional consequence. But Proposition 21 entails that *however* X is chosen, reasonably or unreasonably, Definition 17 will return exactly the symbols in X as constants.

4.2.2 Classical first-order logic

We cannot expect a similar result for first-order logic, already because, as stated in Section 3.2.2, $\models_{FO}$ is not a Bolzano consequence relation. It is, however, a consequence relation (once the language is fixed), so Definition 17 applies. Let us just check that the definition returns the right constants, i.e. that

(19) $\quad C_{\models_{FO}} = X_1 = \{\neg, F_{\langle t,t\rangle}, \wedge, \vee, \forall, \exists, =\}$

But this is fairly clear: First, for each symbol u in X_1 we can easily produce an instance of $\models_{FO}$ that gets destroyed when u is replaced by another symbol of the same category. Take $=$ as an example: we have $a = b \models_{FO} b = a$ but $aRb \not\models_{FO} bRa$, where R is a binary predicate symbol. Second, this cannot be done for any predicate letter or individual constant symbol, precisely because the definition of $\models_{FO}$ universally quantifies of all interpretations of these symbols. $\Box$

4.2.3 Other logics

To repeat, Definition 17 applies to any consequence relation, not just Bolzano consequence relations. For example, consider *intuitionistic propositional logic*, with propositional connectives $X_2 = \{\neg, F_{\langle t,t\rangle}, \wedge, \vee, \rightarrow, \leftrightarrow\}$. Its consequence relation, say $\vdash_{IPL}$, is defined axiomatically,[12] but again it will be clear that

[12] The only unusual feature here is the presence of $F_{\langle t,t\rangle}$, but $F_{\langle t,t\rangle}p$ can be taken to be equivalent to $p \wedge \neg p$. So $\vdash_{IPL}$ is defined via some standard axiomatization of intuitionistic

replacing a symbol in X_2 will destroy some *IPL*-consequence, and moreover that replacing ordinary propositional letters cannot have such an effect. That is,

(20) $\quad C_{\vdash_{IPL}} = X_2$

This is just one example; clearly, similar reasoning can be applied to various other logical systems to show that (with a vocabulary chosen so that Fact 20 is not a problem) the present definition of constancy returns the right set of symbols.

4.3 An isomorphism

The second criterion of the correctness of a method of retrieving constants from a consequence relation mentioned in the Introduction was that it is in some sense inverse to the method of defining consequence from a set of constants. Here we have only partial success. More precisely, we succeed for strongly minimal sets of symbols.

Let *MIN* (*STMIN*) be the set of (strongly) minimal sets of symbols. By Proposition 12, the image of the mapping $\Rightarrow_{-}$ restricted to *MIN* is still all of *BCONS*. Let $STBCONS \subseteq BCONS$ be the image of $\Rightarrow_{-}$ restricted to *STMIN*.

Corollary 22
The mapping $\Rightarrow_{-}$ is an isomorphism between $(STMIN, \subseteq)$ and $(STBCONS, \subseteq)$, whose inverse is C_{-}.

Proof. ¿From Corollary 16 and Fact 19, we see that $\Rightarrow_{-}$ is a bijection with inverse C_{-}. It follows from Lemma 15 that this bijection is an isomorphism. □

5 Further directions

The main purpose of this paper has been introductory: to present some ideas and results in a field of investigation that could be taken further. Various aspects of the framework could be generalized, and variants of the main definitions could be considered. For one thing, working in a Bolzano style substitutional setting leaves you vulnerable to 'accidental' effects of the expressive means of the language. An obvious instance was highlighted in Fact 20. More subtly, the peculiarities illustrated by Examples 1 and 2 in Section 4.1 also seem partly related to the use of a substitutional framework. Some of these issues would disappear, or be significantly altered, if we switch to a Tarski style framework, characterized by the slogan: *Instead of replacing symbols, reinterpret them!*

On the other hand, the Bolzano setting was not chosen merely for simplicity. Especially when you consider consequence relations as primitive, the use of a fixed interpreted language seems quite natural. In fact, whereas going from sets

propositional logic plus this defining axiom.

of constants to consequence relations is a process that transfers fairly smoothly from a substitutional to a model-theoretic setting, it seems less obvious how to effect this transfer when going in the other direction. I believe that both settings, and the relations between them, are worth studying.

In any case, it is clear that more needs to be done, if the goal of viewing the two processes as *inverse* to each other is to be attained. We achieved the strictest kind of inverse relationship — isomorphism — for the case of strongly minimal sets of symbols, but we didn't extend this to arbitrary sets, nor did we provide an independent characterization of the set of consequence relations that are generated from strongly minimal sets.

A weaker inverse relationship can be described as follows: Start from an arbitrary $X \subseteq \mathit{Symb}$, form the consequence relation $\Rightarrow_X$, then extract the constants with C_{-}: Is the relation generated by the set of extracted constants the same as the original one? If so, we have

(21) $\quad \Rightarrow_X \; = \; \Rightarrow_{C_{\Rightarrow_X}}$

even when $C_{\Rightarrow_X}$ is a proper subset of X.

But this fails in general in the present framework. Consider again Example 2 from Section 3.2.3. We have

$$c \in C_{\Rightarrow_{\{a,c\}}}$$

since $\emptyset \Rightarrow_{\{a,c\}} Rbc$ but $\emptyset \not\Rightarrow_{\{a,c\}} Rba$. However, a is not in $C_{\Rightarrow_{\{a,c\}}}$: checking the relevant valid inferences (see (10c)), none becomes invalid when a is replaced by b or by c. So, by Lemma 18,

(22) $\quad C_{\Rightarrow_{\{a,c\}}} = \{c\}$

But we also have

$$\emptyset \Rightarrow_{\{a,c\}} Rab$$

whereas

$$\emptyset \not\Rightarrow_{\{c\}} Rab$$

(by the replacement that permutes a and b). This shows that

$$\Rightarrow_{\{a,c\}} \; \neq \; \Rightarrow_{C_{\Rightarrow_{\{a,c\}}}}$$

so we have a counter-example to (21).

The same example also provides a negative answer to another natural question: although $X \subseteq Y$ always implies $\Rightarrow_X \subseteq \Rightarrow_Y$, what about monotonicity in the other direction, i.e.

(23) $\quad$ If $\Rightarrow \subseteq \Rightarrow'$, then $C_{\Rightarrow} \subseteq C_{\Rightarrow'}$.

Even when the consequence relations are Bolzano consequences, this can fail: we have $\Rightarrow_{\{a\}} \subseteq \Rightarrow_{\{a,c\}}$ in Example 2 but, by (22), and (15) in Section 4.1,

$C_{\Rightarrow_{\{a\}}} \not\subseteq C_{\Rightarrow_{\{a,c\}}}$.

There are moves one can make to insure that (21), and even (23), holds, but a discussion of these, and a fuller account of the various ways of going from consequence relations to constants and back, is deferred to another occasion.[13]

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[13] The extended abstract Bonnay and Westerståhl (2010) outlines some of this.