Constant operators: partial quantifiers

Dag Westerståhl

1. Introduction

This paper concerns a small part of a larger project to do with the notion of *logical constants*. A background idea is that this notion is not necessarily unitary, but contains distinct ingredients. One such ingredient is often called *topic neutrality*, and made precise in the form of requirements of *invariance* under various transformations between models. Such invariance has been the object of several recent studies (Feferman 1999, van Benthem 2002, Bonnay 2006) and is rather well understood. Another ingredient may be called *constancy*; in one version, it is the idea that a logical operator 'means the same' over different universes. It is less clear how this idea should be made precise, and the project mentioned deals with just that.

The goal is to treat operators of any *type*. By "type" we mean type in a simple type theory à la Church. There are several familiar versions of this theory, and in many cases it is just a matter of convenience which version one chooses. It seems, however, that notions of 'constancy' are somewhat sensitive to this choice. For example, does one use *relational* or *functional* types? Also, are they types of *total* or *partial* objects? To begin, at least, one needs to look at what 'constancy' could amount to in each of these cases. In particular, the introduction of partial objects appears to complicate matters. It may be that much of this complication is spurious or unnecessary, but if so, that is something that needs to be established.

First-order generalized quantifiers, or just *quantifiers*, are a paradigmatic kind of operators, well studied in logic as well as language (see, for example, Peters and Westerståhl 2006 for an overview). The issue of which of these operators are logical constants has often been raised: The logician's $\forall$ and $\exists$ are of course prime examples, but what about something like *most* or *each other*? Quantifiers are often presented in a relational framework, but some prefer a functional one, finding that it better reflects the compositional function-argument structure commonly ascribed to the syntax of natural languages. Usually, quantifiers are seen as total, but in their pioneering paper on generalized quantifiers in natural language, Barwise and Cooper (1981), who worked in a functional framework, actually used partial quantifiers; more precisely, quantifiers that were *undefined* for certain arguments.

This paper focuses on one aspect of what 'constancy' could mean for quantifiers in a partial framework, with a view to extend the results to (a) operators of other first-order types (such as the denotations of many adjectives), and eventually, (b) arbitrary types. More precisely, it focuses on the condition of *extension* (EXT), familiar from generalized quantifier theory. Since quantifiers are ubiquitous in logic and language, intuitions about them are stronger than in other cases, and might lead to insights about those cases. Of course, the insights may be negative: perhaps first-order quantifiers (i.e. quantifiers over individuals) form a very special class, and the senses in which they are constant will not transfer to other types. Some such issues will be discussed here.

Apart from Barwise and Cooper (1981), little has been written about partial quantifiers, and Barwise and Cooper did not deal with the issue of 'constancy'. Not did they use the full power of partial quantifiers, as partial functions whose *arguments* may also be partial. The observations in the present paper are preliminary and exploratory. I first present and compare some main versions of simple type theory, and then discuss how quantifiers fare in these systems.

2. Versions of simple type theory

A type theory usually comes with (i) a set of *types*, (ii) a definition of the *objects* of the various types, (iii) a *language* in which one can talk about these objects, and (iv) a *logic* for this language, described both semantically in terms of what its expressions *denote*, and proof-theoretically in terms of a formal deduction system for how certain of these expressions can be *inferred* from others. Here I will only be concerned with the first two of these aspects, and will therefore talk of *type systems* rather than type theories.

The use of type theory in model-theoretic semantics (aspects (i) – (iii) above) began with Montague (see Montague 1974). There are the basic types e of *individuals* and t of *truth values*. Montague wanted to handle *intensional* phenomena and so he had another type s for possible worlds or *indices*, which however was not quite on a par with the other basic types. His type theory, and the accompanying language and intensional logic *IL* did what he wanted them to do, but from a mathematical point of view they had some idiosyncracies (which, for example, resulted in the fact the Church-Rosser theorem did not hold for *IL*). Indeed, Barwise and Cooper (1981) said that Montague Grammar "looks a bit like a Rube Goldberg machine" (p. 204). It was later realized that by using a more standard format of simple type theory, only with an extra basic type s, the idiosyncracies disappear but the usefulness for semantics remains (cf. Gallin 1975, van Benthem and Doets 1983, Muskens 1989a). Another observation was that s played little role in parts of the theory — the extensional part — and in practice most of the theory of quantifiers developed within the Montague framework used only the extensional part.

The type systems to be considered here all have e and (when needed) t as basic types. However, nothing seriously turns on that; everything we say is adaptable to the presence of other basic types.

2.1. Universal operators

Let Θ be a type system where the set T^Θ of types is generated inductively from the basic e and (possibly) t. For each universe M, $M_e^\Theta = M$ and M_t^Θ is a fixed set of truth values (usually $\{\mathrm{T},\mathrm{F}\}$). Further, M generates for each $\tau \in T^\Theta$ (by a definition following the one for the types) the *domain* M_τ^Θ of objects of type τ. (We leave off the superscript $^\Theta$ whenever possible.)

> **Definition**: Let $\tau \in T^\Theta$. A *universal operator in* Θ *of type* τ is a function(al) u that with each universe M associates a unique object u_M in M_τ^Θ.

Various natural language expressions, in particular determiners and noun phrases, are naturally taken to denote universal operators. It is for such operators that the issue of 'constancy', or logicality, is raised.

2.2. Standard functional type systems

Let TFT be the type system whose types are given by

(a1) a basic type (e and t) is a type

(a2) if σ and τ are types, so is $\langle\sigma,\tau\rangle$

and the corresponding objects by (dropping the superscript $^{\mathrm{TFT}}$)

(b1) $M_t = \{\mathrm{T},\mathrm{F}\}$

(b2) $M_e = M$

(b3) $M_{\langle\sigma,\tau\rangle}$ = the set of all total functions from M_σ to M_τ = $[M_\sigma \longrightarrow M_\tau]$

This is probably the most widely used (simple) type system. Sets and relations are rendered as characteristic functions. Note that all functions are unary. To deal with functions of several arguments one uses *currying*, based on the natural bijection between $[X \times Y \longrightarrow Z]$ and $[X \longrightarrow [Y \longrightarrow Z]]$. However, curried types tend to get rather complex, and moreover the bijection fails for *partial* functions (see Muskens 1989b), to be considered later. Therefore, we focus on a variant of TFT that allows multi-argument functions; call this system TFT$^+$:[1]

(c1) a basic type is a type

(c2) If $\sigma_1,\ldots,\sigma_n$ and τ are types, so is $\langle\sigma_1\ldots\sigma_n,\tau\rangle$ $(n \geq 1)$

(d1) $M_e = M$

(d2) $M_t = \{\mathrm{T},\mathrm{F}\}$

(d3) $M_{\langle\sigma_1\ldots\sigma_n,\tau\rangle} \;=\; [M_{\sigma_1} \times \cdots \times M_{\sigma_n} \longrightarrow M_\tau]$

Every type τ in TFT$^+$ can be uniquely written as

$$\tau \;=\; \langle\sigma_{11}\ldots\sigma_{1k_1},\ldots,\langle\sigma_{n1}\ldots\sigma_{nk_n},\tau_0\rangle\ldots\rangle$$

where τ_0 is e or t $(n \geq 0)$. Following the terminology in van Benthem (1989) (for TFT), we call τ *individual* if $\tau_0 = e$ and *Boolean* if $\tau_0 = t$. Boolean types are relations but their arguments need not be; cf. $\langle\langle e,e\rangle,t\rangle$. The *strictly relational* types of TFT$^+$ are defined (inductively) as follows:

> **Definition**: τ is *strictly relational* iff τ is either primitive or of the form $\langle\sigma_1\ldots\sigma_n,t\rangle$, where each σ_i is strictly relational.

(e is included for convenience here.) It is easily seen that τ is strictly relational if and only if no occurrence of a right bracket $\rangle$ in τ is immediately preceded by an occurrence of e or of another right bracket.

We end by noting the following characteristic fact about TFT$^+$; it does not hold for relational or partial types (see below).

Fact 1
If $\tau \neq \tau'$ in TFT$^+$, then $M_\tau \cap M_{\tau'} = \emptyset$, provided the truth values do not belong to any M_σ except M_t.

Proof. Induction. It is clear that

$$\tau \neq e \implies M_\tau \cap M_e = \emptyset$$

$$\tau \neq t \implies M_\tau \cap M_t = \emptyset$$

For the induction step, suppose $f \in M_\tau \cap M_{\tau'}$, where $\tau = \langle \sigma_1 \dots \sigma_n, \tau_0 \rangle$ and $\tau' = \langle \sigma_1' \dots \sigma_n', \tau_0' \rangle$. It follows that

$$dom(f) = M_{\sigma_1} \times \cdots \times M_{\sigma_n} = M_{\sigma_1'} \times \cdots \times M_{\sigma_n'} \neq \emptyset$$

and thus $M_{\sigma_i} = M_{\sigma_i'}$, $1 \leq i \leq n$. By induction hypothesis, $\sigma_i = \sigma_i'$, $1 \leq i \leq n$. Now take $a \in dom(f)$. Then

$$f(a) \in M_{\tau_0} \cap M_{\tau_0'}$$

Again by induction hypothesis, $\tau_0 = \tau_0'$, and so $\tau = \tau'$. □

2.3. Relational types

Although Montague Grammar uses a functional type system, many of its operators, and in particular quantifiers, have an essentially relational character. More exactly, their functional types (in TFT$^+$) are strictly relational in the above sense. Muskens (1989a) concludes from this and other arguments that Montague Grammar is more simply formulated in a framework that is relational by definition. Let RT have the following types and objects:

(a1) The only primitive type is e.

(a2) If $\tau_1, \dots, \tau_n$ are types ($n \geq 0$), so is $(\tau_1, \dots, \tau_n)$.

(b1) $M_e = M$

(b2) $M_{(\tau_1,\dots,\tau_n)} = P(M_{\tau_1} \times \cdots \times M_{\tau_n})$

Here it is stipulated that the cartesian product of the empty sequence () of sets is $\{\emptyset\}$. Thus, $M_{()} = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \{0,1\} = \{\mathrm{F},\mathrm{T}\}$, so () corresponds to the type t in TFT.

It is not hard to establish that there is an isomorphism between RT and the strictly relational part of TFT$^+$ (cf. Muskens 1989b):

Fact 2
The mapping π from the RT-types to strictly relational TFT$^+$-types given by

$$\begin{cases} \pi(e) & = \ e \\ \pi(()) & = \ t \\ \pi((\tau_1,\ldots,\tau_n)) & = \ \langle \pi(\tau_1)\ldots\pi(\tau_n),t\rangle \quad (n \geq 1) \end{cases}$$

is a bijection. Moreover, it extends naturally to bijections π_M from M^{RT}_{τ} to $M^{TFT^+}_{\pi(\tau)}$, in such a way that for $R \in M^{RT}_{(\tau_1,\ldots,\tau_n)}$ and $a_i \in M^{RT}_{\tau_i}$,

$$(a_1,\ldots,a_n) \in R \iff \pi_M(R)(\pi_M(a_1),\ldots,\pi_M(a_1)) = \mathrm{T}$$

Accordingly, π further extends to a bijection from universal operators u in RT to universal operators in TFT$^+$ over strictly relational types, letting

$$\pi(u)_M \ = \ \pi_M(u_M)$$

The mappings π_M should really be indexed for types as well, writing $\pi_{M,\tau}$, but we leave that index out for perspicuity. To see the need for it, note that the domains in RT are *not* in general disjoint; for example $\emptyset$ belongs to all M^{RT}_{τ} for $\tau \neq e$, $\{\emptyset\}$ belongs to $M^{\mathrm{RT}}_{((e))}$, $M^{\mathrm{RT}}_{(((e)))}$, etc. So π_M maps the empty set to different objects depending on which type we are considering; more exactly, to the 'empty' characteristic function in that type. But this does not prevent each $\pi_{M,\tau}$ from being a bijection.

2.4. Going partial

Muskens (1989a) argues that partiality naturally belongs to a formal semantical framework; the most obvious reasons coming from apparent truth value gaps and lack of denotation of certain terms. There are also purely conceptual advantages.

For one example, take the denotation of the predicate *prime* (number). Arguably, only natural numbers can be prime or non-prime; there is something amiss with asking, for example, if the dog Fido is prime. We could of course introduce a new basic type for the natural numbers, but if we prefer to stick with e and t (while treating treat numbers as individuals, not sets), another option is to stipulate that the set $\mathbb{N}$ of numbers is the 'range of significance' of the predicate *prime*, or, more precisely, that in each universe of discourse M, that range is the set of numbers *in* M.

Similarly, consider the *successor function*, $S(n) = n+1$, taken to be of type $\langle e, e\rangle$. This function is only defined for natural numbers, so in an arbitrary universe M, S_M is naturally taken to be a partial function. Its domain is not quite the set of numbers in M; rather

$$dom(S_M) = \{n \in M \cap \mathbb{N} : n+1 \in M\}$$

(Not only the arguments but also the values must belong to M.)

How can we modify a type system to take care of partiality? In RT, it is fairly clear what to do. A predicate generally gets a *positive* and a *negative* extension, i.e. two sets P^+ and P^- of objects of the appropriate type. Their union is the range of significance; in the example above, P^+ is the set of primes in that range, and P^- the set of non-primes; Fido belongs to neither.

In full generality, let (following Muskens 1989a) a *partial relation* be a pair of ordinary relations, the first member of which is its *positive part*, or *extension*, and the second its *negative part*, or *anti-extension*. In the corresponding type system, which we call PRT, types are as in RT but the objects of each type are defined as follows:

(a1) $M_e = M$

(a2) $M_{(\tau_1,\ldots,\tau_n)} = (P(M_{\tau_1} \times \cdots \times M_{\tau_n}))^2$

A partial relation in $M_{(\tau_1,\ldots,\tau_n)}$ is *coherent* iff its negative and positive parts are disjoint, and *classical* iff its negative part is the complement of its positive part. Let PRT-3 be the version of PRT which constrains partial relations to be coherent, and PRT-2 the one which constrains them to be classical (the choice of labels will become clear below). RT is clearly isomorphic to PRT-2.

For the functional type systems, there are essentially two ways to 'go partial': either consider partial functions instead of total ones, or keep total functions but add an extra truth value. Modifying TFT$^+$ in the first way is extremely simple: just take the set of all partial functions from X to Y instead, which we will denote $[X \hookrightarrow Y]$. That is, PFT$^+$ is the type system which has the same types as TFT$^+$, and whose objects are given by:

(c1) $M_e = M$

(c2) $M_t = \{\mathrm{T}, \mathrm{F}\}$

(c3) $M_{\langle\sigma_1\ldots\sigma_n,\tau\rangle} \;=\; [M_{\sigma_1} \times \cdots \times M_{\sigma_n} \hookrightarrow M_\tau]$

This will be our main candidate here for a partial version of functional type theory.[2]

Note that distinct domains in PFT$^+$ are not disjoint, since every non-primitive type now contains a *null object*: the function (of the type in question) with empty domain, which can be identified with the empty set. This must of course not be confused with the various (total) characteristic functions of the empty set.

Fact 3
For all τ, $M_\tau^{TFT^+} \subseteq M_\tau^{PFT^+}$.

Proof. Induction. The base step is immediate. Consider $\langle\sigma_1 \ldots \sigma_n, \tau\rangle$ and suppose $M_{\sigma_i}^{\mathrm{TFT}^+} \subseteq M_{\sigma_i}^{\mathrm{PFT}^+}$, $1 \le i \le n$, and $M_\tau^{\mathrm{TFT}^+} \subseteq M_\tau^{\mathrm{PFT}^+}$. An object F in $M_{\langle\sigma_1\ldots\sigma_n,\tau\rangle}^{\mathrm{TFT}^+}$ is a *total* function from $M_{\sigma_1}^{\mathrm{TFT}^+} \times \cdots \times M_{\sigma_n}^{\mathrm{TFT}^+}$ to $M_\tau^{\mathrm{TFT}^+}$. Thus, F is a *partial* function from $M_{\sigma_1}^{\mathrm{PFT}^+} \times \cdots \times M_{\sigma_n}^{\mathrm{PFT}^+}$ to $M_\tau^{\mathrm{PFT}^+}$ (cf. note 2). $\square$

Corollary 4
Every universal operator in TFT$^+$ is also a universal operator in PFT$^+$.

We may now expect PRT to correspond to the strictly relational part of PFT$^+$. Before looking at this, however, we should consider the other way of partializing TFT$^+$, i.e. by introducing extra truth values. Actually, there are two ways to go about this. Let TFT$_3^+$ be just as TFT$^+$ except that

$$M_t = \{\mathrm{T}, \mathrm{F}, \mathrm{N}\}$$

and let TFT$_4^+$ be like TFT$^+$ except that

$$M_t = \{\mathrm{T}, \mathrm{F}, \mathrm{N}, \mathrm{B}\}$$

(The notation is from Belnap 1977; ‘N’ stands for ‘neither’ and ‘B’ for ‘both’.)

The main idea, of course, is that instead of saying that a function F with values in M_t is *undefined* for a certain argument a, we stipulate that $F(a) =$ N (or $F(a) =$ B). Note that this works only for functions with values in M_t.

More generally, it extends to the strictly relational part of TFT^+, but not to a type like $\langle e, e\rangle$. But what about the fourth truth value?

TFT_3^+ and TFT_4^+ essentially commit us to a 3-valued or 4-valued logic, respectively. It turns out that for many purposes, the 4-valued version is quite natural, and mathematically more elegant. This can be seen already from PRT. In the general case, a partial relation splits the domain into *four* parts; only if we add the coherence requirement do we get three.[3]

Muskens (1989a) gives a clear and informative account of 3- and 4-valued logic, and the generalization to type theory. In particular, he shows that, as one would expect, there is a tight connection between PRT and the (strictly) relational part of TFT_4^+.[4]

It should be fairly obvious by now how to map (isomorphically) the objects in M_τ^{PRT} to the the objects in $M_{\pi(\tau)}^{\mathrm{TFT}_4^+}$; we omit the details. Likewise, one sees how PRT-3 maps isomorphically onto the strictly relational part of TFT_3^+.

Note also that Fact 1 holds (with the same proof) for TFT_3^+ and TFT_4^+.

Now, what about partial sets and relations in PFT^+? If we restrict attention to *coherent* relations, they are already there; more exactly, their *partial characteristic functions* are. The positive part of such a relation is mapped to T, the negative part to F, and the mapping is *undefined* on the remaining part. It is easy to see that in this way one obtains an isomorphism between PRT-3 and the strictly relational part of TFT_3^+. Similarly, the strictly relational part of TFT_3^+ and PFT^+ are isomorphic; we omit details.

PFT^+ admits a 3-valued logic but doesn't necessitate one. Of course one needs to make a decision about what to make of a sentence "c is P" when c denotes an object for which the characteristic function denoted by P is undefined. Farmer (1990) stipulates that the sentence is false in this case (thus preserving a 2-valued logic), arguing that this fits best with *mathematical* practice.[5] Since our concern is type systems rather than type theory, we shall not pursue this further here.

2.5. Summing up

We have looked at a variety of similar but not equivalent formulations of simple type theory. Which of these one prefers can be a matter of usefulness for the purpose at hand, but also a matter of taste. In the context of the present paper, the following considerations can be made:

1. We are eventually interested not only in the (strictly) relational types but also in the purely functional types, such as $\langle e,e\rangle$.
2. As to partial relations, the coherence constraint seems natural.
3. One main source of intuition are generalized quantifiers, which are relational.

Therefore, among partial systems we focus on PFT$^+$, and among total ones, RT and TFT$^+$.

3. Monotonicity and related properties

Let Θ be a type system of the kind considered here. On issue that turns out to be important to the matter of 'constancy' (but is rarely considered in the literature, as far as I know), is to what extent the domains of a given type τ can *overlap* for different universes of individuals. In particular, if you extend the universe (of discourse), do you then also extend the corresponding domain M_τ^Θ? If you do, we say that τ is *monotone*. The main result in this section is a characterization of the monotone types for each of the type systems in section 2. We use the following terminology.

> **Definition**: A type τ in a type system Θ is
> (a) *monotone* iff $M \subseteq M'$ implies $M_\tau^\Theta \subseteq M_\tau'^\Theta$;
> (b) *distinct* iff $M \neq M'$ implies $M_\tau^\Theta \neq M_\tau'^\Theta$;
> (c) *disjoint* iff $M \neq M'$ implies $M_\tau^\Theta \cap M_\tau'^\Theta = \emptyset$.

For example, we have seen that types in RT are not disjoint, since each domain (except M_e) contains the empty set. We need one more

> **Definition**: A type τ in a type system Θ is *truth-functional* if (the expression) τ does not contain e.

The domain of a truth-functional type does not depend on M, so these types are trivially monotone and non-distinct.

Theorem 5 *All types in RT, PRT, PRT-3, and PFT$^+$ are monotone. All non-truth-functional types in these systems are distinct.*

Proof. Use induction. Obviously, t and e are monotone. Among the relational systems, consider PRT (the others are similar). Suppose $M \subseteq M'$. If (dropping the superscript) $M_{\tau_i} \subseteq M'_{\tau_i}$, $1 \leq i \leq n$, and $(R_1, R_2) \in M_{(\tau_1,\ldots,\tau_n)}$, then

$$R_j \subseteq M_{\tau_1} \times \cdots \times M_{\tau_n} \subseteq M'_{\tau_1} \times \cdots \times M'_{\tau_n}$$

$j = 1,2$. Thus $(R_1, R_2) \in M'_{(\tau_1,\ldots,\tau_n)}$.

For PFT$^+$, consider $\langle \sigma_1 \ldots \sigma_n, \tau \rangle$. $F \in M_{\langle \sigma_1 \ldots \sigma_n, \tau \rangle}$ is a partial function from $M_{\sigma_1} \times \cdots \times M_{\sigma_n}$ to M_τ. If $M \subseteq M'$ then, by induction hypothesis, $M_{\sigma_i} \subseteq M'_{\sigma_i}$, $1 \leq i \leq n$, and $M_\tau \subseteq M'_\tau$. Therefore, F is a partial function from $M'_{\sigma_1} \times \cdots \times M'_{\sigma_n}$ to M'_τ.

This proves the first claim of the theorem. For the second claim, consider again the relational type systems first. e is trivially distinct, and it is clearly enough to show that

$$M_{(\sigma_1,\ldots,\sigma_n)} = M'_{(\sigma_1,\ldots,\sigma_n)} \text{ implies that } M_{\sigma_i} = M'_{\sigma_i} \text{ for } 1 \leq i \leq n$$

for all RT-types $\sigma_1, \ldots, \sigma_n$. Consider PRT; the other cases are similar. Fix i between 1 and n and suppose $(R_{i1}, R_{i2}) \in M_{\sigma_i}$. Take any $(R_{j1}, R_{j2}) \in M_{\sigma_j}$ for $1 \leq j \leq n$, $j \neq i$. Then

$$((R_{11}, R_{12}), \ldots, (R_{n1}, R_{n2})) \in M_{\sigma_1} \times \cdots \times M_{\sigma_n}$$

Thus

$$\{((R_{11}, R_{12}), \ldots, (R_{n1}, R_{n2}))\} \in M_{(\sigma_1,\ldots,\sigma_n)} = M'_{(\sigma_1,\ldots,\sigma_n)}$$

It follows that $(R_{i1}, R_{i2}) \in M'_{\sigma_i}$. This shows that $M_{\sigma_i} \subseteq M'_{\sigma_i}$, and by a symmetric argument we see that $M'_{\sigma_i} \subseteq M_{\sigma_i}$.

For PFT$^+$ one shows instead that

$$M_{\langle \sigma_1 \ldots \sigma_n, \tau \rangle} = M'_{\langle \sigma_1 \ldots \sigma_n, \tau \rangle} \text{ implies } M_{\sigma_i} = M'_{\sigma_i} \text{ for } 1 \leq i \leq n, \text{ and } M_\tau = M'_\tau.$$

The proof of this is similar. □

This result fails for PRT-2 (since the complement of a relation increases when the universe is extended). But that is an artifact of attempting to present RT in the format of PRT. In practice one would use RT instead.

The situation as regards monotonicity is very different in the total functional type systems. We need one final

Definition: A TFT$^+$-type τ is *extended truth-functional* if it is either truth-functional or of the form

(1) $\langle \sigma_{11} \ldots \sigma_{1k_1}, \ldots, \langle \sigma_{n1} \ldots \sigma_{nk_n}, e\rangle \ldots \rangle$

with each σ_{ij} truth-functional. (This includes the case $n = 0$, i.e. $\tau = e$.)

Theorem 6 *The monotone types in TFT$^+$, TFT$_3^+$, and TFT$_4^+$ are exactly the extended truth-functional types. All other types are disjoint. The same holds for TFT and its 3- and 4-valued variants.*

Proof. The claim for TFT and its variants follows from the one for TFT$^+$ and its variants by restricting attention to TFT-types, so we can focus on types in TFT$^+$. The following argument works for any one of TFT$^+$, TFT$_3^+$, and TFT$_4^+$. Clearly, a truth-functional type τ is monotone, since then M_τ is independent of M. Suppose τ has the form (1), where each σ_{ij} is truth-functional. An element F of M_τ can be seen (using currying!) as a function from the product

$$M_{\sigma_{11}} \times \cdots \times M_{\sigma_{nk_n}}$$

to M. If $M \subseteq M'$, then $M_{\sigma_{ij}} = M'_{\sigma_{ij}}$ for each i, j, since σ_{ij} is truth-functional, and it follows that $F \in M'_\tau$. So τ is monotone, and we have verified the first part of the claim. To prove second part, we start with a number of observations:

(2) If at least one of the τ_i is distinct, then $\langle \tau_1 \ldots \tau_n, \sigma\rangle$ is disjoint for any type σ.

The assumption entails that if $M \neq M'$, then $M_{\tau_1} \times \cdots \times M_{\tau_n} \neq M'_{\tau_1} \times \cdots \times M'_{\tau_n}$. But then, if $f \in M_{\langle \tau_1 \ldots \tau_n, \sigma\rangle}$ and $g \in M'_{\langle \tau_1 \ldots \tau_n, \sigma\rangle}$, $dom(f) \neq dom(g)$, so $f \neq g$.

(3) If τ is distinct, then $\langle \sigma_1 \ldots \sigma_n, \tau\rangle$ is distinct for any types $\sigma_1, \ldots, \sigma_n$.

To see this, take $b \in M_\tau - M'_\tau$. There is some $f \in M_{\langle \sigma_1 \ldots \sigma_n, \tau\rangle}$ such that for some $a \in M_{\sigma_1} \times \cdots \times M_{\sigma_n}$, $f(a) = b$. Then $f \notin M'_{\langle \sigma_1 \ldots \sigma_n, \tau\rangle}$.

(4) If τ is disjoint, then $\langle \sigma_1 \ldots \sigma_n, \tau\rangle$ is disjoint for any types $\sigma_1, \ldots, \sigma_n$.

This is because a function in $M_{\langle \sigma_1 \ldots \sigma_n, \tau\rangle}$ and a function in $M'_{\langle \sigma_1 \ldots \sigma_n, \tau\rangle}$ cannot have common values (since τ is disjoint), and so cannot be identical.

(5) τ is distinct if and only if it is not truth-functional.

Proof: Clearly truth-functional types are not distinct. In the other direction, use induction over τ. If $\tau = e$ it is distinct. Suppose $\tau = \langle \sigma_1 \ldots \sigma_n, \tau_0\rangle$. Since

τ is assumed not to be truth-functional, at least one of $\sigma_1 \dots \sigma_n$ and τ_0 is not truth-functional, and hence distinct, by induction hypothesis. But then it follows from (2) and (3) that τ is distinct. This proves (5).

Now we can prove the second part of the proposition. Every type τ has the form

$$\langle \sigma_{11} \dots \sigma_{1k_1}, \dots, \langle \sigma_{n1} \dots \sigma_{nk_n}, \tau_0 \rangle \dots \rangle$$

where τ_0 is either e or t. If each σ_{ij} is truth-functional, then τ is extended truth-functional, and hence monotone by the first part of the proof. Suppose instead that some σ_{ij} is not truth-functional, and hence distinct, by (5). Then

$$\sigma = \langle \sigma_{i1} \dots \sigma_{ik_i}, \dots, \langle \sigma_{n1} \dots \sigma_{nk_n}, \tau_0 \rangle \dots \rangle$$

is disjoint by (2). Therefore, $\langle \sigma_{i-11} \dots \sigma_{i-1k_{i-1}}, \sigma \rangle$ is disjoint by (4). Repeating this argument, it follows that τ is disjoint. This completes the proof. $\square$

4. Ext for standard quantifiers and beyond

The quantifiers we consider are first-order in the sense that they quantify over individuals (*not* in the sense of being definable in first-order logic!). More generally, let the *level* of a (functional or relational) type be the maximal number of *nestings* of angle brackets, or parentheses, that occur in it. Then quantifiers have level 2. Lindström (1966) introduced a practical type system tailor-made for quantifiers, which is still normally used in this context, but here we shall stick to relational or functional types as before. Thus, let the *first-order relational types* be those of the form $\langle e \dots e, t \rangle$ (in TFT$^+$ or PFT$^+$) or $(e \dots e)$ (in RT). A *quantifier* is then a universal operator of type $\langle \sigma_1 \dots \sigma_n, t \rangle$, or $(\sigma_1 \dots \sigma_n)$, where each σ_i is a first-order relational type (in the respective system).

For standard quantifiers, there is a familiar notion of 'constancy' over varying universes, usually called *extension* or Ext. It says that if you extend M to M', the quantifier remains the same on arguments over M. For example, it rules out a quantifier meaning *some* on universes with less that 10 elements and *every* on other universes. In fact, Ext is easily defined for arbitrary types in RT:

Definition: A universal operator u of type $(\sigma_1, \dots, \sigma_n)$ is Ext if $M \subseteq M'$ implies that $u_M = u_{M'} \restriction M \quad [= u_{M'} \cap (M_{\sigma_1} \times \dots \times M_{\sigma_n})]$.

This works because RT-types are monotone. Since RT is isomorphic to the strictly relational part of TFT^+, EXT is defined for arbitrary strictly relational types in TFT^+ as well. But note that in TFT^+ it is almost *never* the case that $u_M = u_{M'} \upharpoonright M$, in the sense of ordinary function restriction, since the arguments of u_M are functions with domain M (when u is a quantifier), whereas $u_{M'}$ has arguments with domain M' (cf. Fact 1). The reformulation of EXT for strictly relational types becomes clumsier in TFT^+, precisely because types in TFT^+ are not monotone.

Eventually we want to generalize in three directions from the case of quantifiers: (1) to arbitrary (strictly) relational types (for EXT this was done above), (2) to other functional types, and (3) to partial types. It is then convenient to take the functional type systems as a starting-point, even when dealing with quantifiers. If we need to go back to the relational case (to RT) we use the mapping π from Fact 2.

Here our focus is on (3). Remaining with the types of quantifiers, we consider the corresponding partial objects. That is, we look at universal operators of these types in PFT^+, or in other words, *partial quantifiers*.

5. Partial quantifiers

A partial quantifier q of type $\langle \sigma_1 \ldots \sigma_n, t\rangle$ may exhibit partiality in two ways: q_M may itself be a partial (characteristic) function, and it may take partial (characteristic) functions as arguments, i.e. the domain of q_M could be the whole $M^{\mathrm{PFT}^+}_{\sigma_1} \times \cdots \times M^{\mathrm{PFT}^+}_{\sigma_n}$ (or any subset of it). As noted, Barwise and Cooper (1981) considered the first kind of partiality. A quantifier like *the three* was only defined for those (characteristic functions of) subsets of M having exactly three elements. The intuition was that a sentence like *The three boys failed the exam* has no truth value unless there are exactly three boys in the (discourse) universe. For now, however, we must allow both kinds of partiality. To simplify notation we often consider the case of type $\langle ee, t\rangle$ below (type $((e,e))$ in RT, or $\langle 2\rangle$ in the standard typing of quantifiers), but everything we say generalizes to arbitrary quantifier types (and some of it to arbitrary strictly relational types).

Even if our main objects are characteristic functions, it is convenient to have a notation for the sets they correspond to. The following definition is formulated for the case of $\langle ee, t\rangle$ but works for any first-order relational type. For $f \in M^{\mathrm{PFT}^+}_{\langle ee,t\rangle}$, let

(6) $f^{+M} = \{(a,b) \in M^2 : f(a,b) = \mathrm{T}\}$

(7) $f^{-M} = \{(a,b) \in M^2 : f(a,b) = \mathrm{F}\}$

f^{+M} and f^{-M} together determine f. That is, we have the following

Fact 7
If $f^{+M} = g^{+M'}$ and $f^{-M} = g^{-M'}$, then $f = g$ (and hence $dom(f) \subseteq (M \cap M')^2$).

Experience with partial quantifiers in the full sense seems limited, but one obvious source is 'partial versions' of already familiar total quantifiers.

5.1. Total quantifiers as partial quantifiers

By Corollary 4, an ordinary total quantifier Q (in TFT$^+$) already *is* a quantifier in PFT$^+$. The switch to PFT$^+$ is a change of perspective on the *same* object. What does this switch amount to for EXT?

Roughly, EXT in TFT$^+$ says that, for arguments f of Q_M, the universe outside f^{+M} is irrelevant. Using the *same* condition in PFT$^+$ we get the following property:

> **Definition**: q is p-EXT$^{\mathrm{gq,s}}$ if, whenever $f \in dom(q_M)$, $g \in dom(q_{M'})$, and $f^{+M} = g^{+M'}$, it holds that $q_M(f) = q_{M'}(g)$.[6]

(p' stands for 'partial', 'gq' for 'generalized quantifier', and 's' for strong; cf. below.) One easily sees that we have:

Fact 8
A quantifier in TFT$^+$ is EXT *if and only if it is p-*EXT^{gq,s} *as an operator in PFT$^+$.*

However, in the partial case it seems at least as natural to consider the weaker condition that everything outside f^{+M} *and* f^{-M} is irrelevant:

> **Definition**: q is p-EXT$^{\mathrm{gq}}$ if, whenever $f \in dom(q_M)$, $g \in dom(q_{M'})$, $f^{+M} = g^{+M'}$, and $f^{-M} = g^{-M'}$, it holds that $q_M(f) = q_{M'}(g)$.

Using the fact that f^{+M} *and* f^{-M} determine f we can find a simpler formulation of this requirement:

Fact 9
q is p-$\text{E}\scriptstyle{\text{XT}}^{gq}$ *if and only if it has the following property:*

(8) *If $f \in dom(q_M) \cap dom(q_{M'})$ and $M \subseteq M'$, then $q_M(f) = q_{M'}(f)$.*

Proof. That (8) follows from p-$\text{E}\scriptstyle{\text{XT}}^{gq}$ is immediate. In the other direction, given the assumptions in p-$\text{E}\scriptstyle{\text{XT}}^{gq}$, use Fact 7, and apply (8) to M and $M \cup M'$, and to M' and $M \cup M'$. □

(8) no longer mentions f^{+M} or f^{-M}. It is thus a putative formulation of EXT for arbitrary types in PFT$^+$. Indeed, it is a slight strengthening of what is perhaps the most obvious candidate for EXT in a partial functional framework:

> **Definition** A universal operator u of type $\langle \sigma, \tau \rangle$ in PFT$^+$ is p-EXT if $M \subseteq M'$ entails $u_M = u_{M'} \upharpoonright M$.[7]

p-EXT is stronger than (8), since $u_M = u_{M'} \upharpoonright M$ entails that $dom(u_M) \subseteq dom(u_{M'})$, which does not follow from (8). As an example, consider, the successor function S mentioned earlier, taken as a universal operator of type $\langle e, e \rangle$, such that for each M, $dom(S_M) = \{n \in M \cap \mathbb{N} : n+1 \in M\}$. S is clearly p-EXT.

But we have already observed that a condition like p-EXT can never hold in a total functional framework. More precisely, we have:

Fact 10
For any universal operator in TFT$^+$ of type $\langle\langle \tau_1 \tau_2, t \rangle, t \rangle$ (where τ_1, τ_2 are strictly relational), when seen as an operator in PFT$^+$, p-$\text{E}\scriptstyle{\text{XT}}^{gq}$ *trivially holds. Also, p-*EXT *trivially fails, except when the type is truth-functional.*

Proof. Suppose $M \subseteq M'$. If Q is such an operator, $dom(Q_M) = M^{\text{TFT}^+}_{\langle \tau_1 \tau_2, t \rangle}$, and $dom(Q_{M'}) = M'^{\text{TFT}^+}_{\langle \tau_1 \tau_2, t \rangle}$. Now consider two cases.

Case 1: At least one of τ_1 and τ_2 is not truth-functional. Then, since both are strictly relational, it follows that the type $\langle \tau_1 \tau_2, t \rangle$ is not extended truth-functional. Hence, by Theorem 6, if $f \in dom(Q_M) \cap dom(Q_{M'})$, then $M = M'$, and p-$\text{E}\scriptstyle{\text{XT}}^{gq}$ holds trivially. Similarly, if M is a proper subset of M', it can never hold that $u_M = u_{M'} \upharpoonright M$, so p-EXT fails.

Case 2: Both τ_1 and τ_2 are truth-functional. Then the arguments of Q_M do not depend on M at all, so p-$\text{E}\scriptstyle{\text{XT}}^{gq}$ again holds trivially (use the formulation (8)), and so does p-EXT. □

It follows that we shall never get any interesting examples of p-EXT^{gq}, but not p-EXT^{gq,s}, quantifiers if we restrict attention to quantifiers in TFT$^+$. But perhaps the conclusion we should draw from this is that even though every total quantifier Q is also a partial quantifier, it isn't really Q itself that is its closest 'partial version'. There are indeed other such versions, which *extend* Q to a partial quantifier.

5.2. Partial version of total quantifiers

A total quantifier Q of type $\langle\langle ee,t\rangle,t\rangle$, say, puts, for any M, a condition on (characteristic functions of) relations $R \subseteq M^2$, or, more exactly, on R and its complement. An obvious way to extend Q to a partial quantifier is to put the *same* condition on relations corresponding to partial characteristic functions f. But now we have a choice whether to take the complement with respect to the whole M^2, or with respect to the possibly smaller set $dom(f)$. Note that in both cases, there seems to be no obvious reason to limit attention to some partial characteristic functions and not to others. Thus, the resulting partial quantifiers will have all of $M^{\mathrm{PFT}^+}_{\langle ee,t\rangle}$ as their domain, for each M.

Let us formulate this precisely, generalizing to the case of a universal operator Q in TFT$^+$ of type $\langle\langle\tau_1\tau_2,t\rangle,t\rangle$, where τ_1 and τ_2 are strictly relational. It is somewhat more perspicuous to use the RT-framework here. Letting $\sigma_i = \pi^{-1}(\tau_i)$, $i = 1,2$, $\pi^{-1}(Q)$ is thus of RT-type $((\sigma_1,\sigma_2))$.

Corresponding to Q we define a global binary relation $\mathbf{R}_Q$ between relations between objects of type σ_1 and objects of type σ_2, as follows:

Definition: For any relations S,R of the above kind:
$\mathbf{R}_Q(S,R) \Leftrightarrow \exists M[R \subseteq M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2} \;\&\; S = (M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2}) - R \;\&\; \pi^{-1}(Q)_M(R)]$

Fact 11
For all M and all $R \subseteq M^{RT}_{\sigma_1} \times M^{RT}_{\sigma_2}$,

$$\pi^{-1}(Q)_M(R) \iff \mathbf{R}_Q((M^{RT}_{\sigma_1} \times M^{RT}_{\sigma_2}) - R, R)$$

Moreover, every such global relation $\mathbf{R}$ *corresponds to a universal operator* $Q_{\mathbf{R}}$ *in TFT*$^+$ *such that* $\mathbf{R}_{Q_{\mathbf{R}}} = \mathbf{R}$.

Proof. We proof the first part and leave the second to the reader. The left-to-right direction is obvious from the definition of $\mathbf{R}$. In the other direction, take $R \subseteq M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2}$ and suppose the right-hand side of the claim holds. Then there is a universe M' such that $R \subseteq M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}$, and $(M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2}) - R = (M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}) - R$, and $\pi^{-1}(Q)_{M'}(R)$.

Now take $(a,b) \in M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2}$. If $(a,b) \in R$, then $(a,b) \in M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}$ by the above. If $(a,b) \in (M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2}) - R$, then again $(a,b) \in M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}$. Thus, $M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2} \subseteq M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}$, and from a similar argument we conclude that $M^{\mathrm{RT}}_{\sigma_1} \times M^{\mathrm{RT}}_{\sigma_2} = M'^{\mathrm{RT}}_{\sigma_1} \times M'^{\mathrm{RT}}_{\sigma_2}$. Thus, $M^{\mathrm{RT}}_{\sigma_1} = M'^{\mathrm{RT}}_{\sigma_1}$, and it then follows from Theorem 5 that $M = M'$. Hence, $\pi^{-1}(Q)_M(R)$. □

So these global relations are just another way to present universal operators of this type, and, generalizing, any total operator of strictly relational type; in particular, any (total) quantifier. We use this presentation to extend from total to partial operators, going back, however, to the case of a quantifier Q of type $\langle\langle ee,t\rangle,t\rangle$ in TFT$^+$.

Definition

Define two quantifiers in PFT$^+$ of the same type, $Q^{p1}$and Q^{p2}, by letting, for each M and each $f \in M^{\mathrm{PFT}^+}_{\langle ee,t\rangle}$,

$$Q^{p1}_M(f) = \mathrm{T} \text{ iff } \mathbf{R}_Q(M^2 - f^{+M}, f^{+M})$$
$$Q^{p2}_M(f) = \mathrm{T} \text{ iff } \mathbf{R}_Q(dom(f) - f^{+M}, f^{+M}) \text{ iff } \mathbf{R}_Q(f^{-M}, f^{+M})$$

This is taken to mean that when the right-hand side is false for f, the operators get the value F.

By Fact 11, it is clear what Q^{p1} means, given Q. The effect of Q^{p2} is the following:

Fact 12
For all $f \in M^{PFT^+}_{\langle ee,t\rangle}$, $Q^{p2}_M(f) = T \Longleftrightarrow \exists A \subseteq M[dom(f) = A^2 \;\&\; \pi^{-1}(Q)_A(f^{+M})]$

Proof. By Fact 11, we see that $Q^{p2}_M(f) = \mathrm{T}$ iff there is a set A such that $f^{+M} \subseteq A^2$, $A^2 - f^{+M} = dom(f) - f^{+M}$, and $\pi^{-1}(Q)_A(f^{+M})$. Therefore, it is enough to show that the right-hand side entails that $dom(f) = A^2$. If $(a,b) \in dom(f)$, then either $f(a,b) = \mathrm{T}$ or $f(a,b) = \mathrm{F}$, so $(a,b) \in f^{+M}$ or $(a,b) \in f^{-M} = dom(f) - f^{+M}$, and in both cases it follows that $(a,b) \in A^2$. In the other direction, if $(a,b) \in A^2$, then $(a,b) \in f^{+M}$ or $(a,b) \in A^2 - f^{+M}$, and in both cases we have $(a,b) \in dom(f)$. □

To see some examples, consider first the simplest type of a quantifier, $\langle\langle e,t\rangle,t\rangle$ (type $\langle 1\rangle$ in the standard notation). For $Q = \exists_{\geq 2}$ we have, when $f \in M^{\mathrm{PFT}^+}_{\langle e,t\rangle}$,

$$(\exists^{p1}_{\geq 2})_M(f) = \mathrm{T} \iff |f^{+M}| \geq 2 \iff (\exists^{p2}_{\geq 2})_M(f) = \mathrm{T}$$

This quantifier just says something about the size of the set f^{+M}, so the complement doesn't matter, and the two partial versions of $\exists_{\geq 2}$ coincide. The reason is that $\exists_{\geq 2}$ is EXT. The partial version is p-EXT$^{\mathrm{gq,s}}$. On the other hand, take the Rescher quantifier Q^R, which is not EXT:

$$(Q^R)^{p1}_M(f) = \mathrm{T} \iff |f^{+M}| > |M - f^{+M}|$$
$$(Q^R)^{p2}_M(f) = \mathrm{T} \iff |f^{+M}| > |f^{-M}|$$

It is clear that $(Q^R)^{p2}$ is p-EXT$^{\mathrm{gq}}$, but not p-EXT$^{\mathrm{gq,s}}$, whereas $(Q^R)^{p1}$ is not even p-EXT$^{\mathrm{gq}}$. These observations are instances of the next fact. Let Q as before be of type $\langle\langle ee,t\rangle,t\rangle$.

Fact 13

(a) *For* $f \in M^{TFT^+}_{\langle ee,t\rangle}$, $Q_M(f) = Q^{p1}_M(f) = Q^{p2}_M(f)$.

(b) *If* Q *is* EXT, *then* $Q^{p1} = Q^{p2}$.

(c) Q *is* EXT *iff* Q^{p1} *is* p-EXT^{gq} *iff* Q *is* p-EXT^{gq,s}.

(d) Q^{p2} *is always* p-EXT^{gq}.

Proof. Straightforward verification, observing that EXT for Q means that the first argument of $\mathbf{R}_Q$ is irrelevant, and that the second part of (c) is Fact 8. □

(a) says that on total (characteristic functions of) relations, Q coincides with its partial versions, so it makes sense to call these *extensions* of Q.

Two examples of type $\langle\langle ee,t\rangle,t\rangle$ are

$$W_M(f) = \mathrm{T} \iff f^{+M} \text{ is a well-ordering of } M$$

$$Wf_M(f) = \mathrm{T} \iff f^{+M} \text{ is a well-founded relation}$$

W is not EXT, since it requires the ordering to be total ($\forall x \exists y P(x,y)$ must be true in the model (M, f^{+M})), but *Wf*, which only says that f^{+M} has no infinite descending chain, is EXT. So there is just one partial version of *Wf*, but the two partial versions of W are distinct. Using Fact 12 we see that they are:

$$W^{p1}_M(f) = \mathrm{T} \Leftrightarrow f^{+M} \text{ is a well-ordering of } M$$
$$W^{p2}_M(f) = \mathrm{T} \Leftrightarrow \exists A \subseteq M[dom(f) = A^2 \ \&\ f^{+M} \text{ is a well-ordering of } A]$$

for $f \in M^{\mathrm{PFT}^+}_{\langle ee,t\rangle}$.

There is a general conclusion to be drawn here, not so much about EXT as about this way of quantifying over partial sets and relations. To express it, we do two things. First, we get rid of the existential quantifier in Fact 12. Indeed, the set A there is determined by f, via the usual *dom* function. To be precise, define this function as follows. Let R be any n-ary relation.

$$dom(R) = R, \text{ if } n = 1$$
$$dom(R) = \{a : \exists b_1,\ldots,b_{n-1} R(a,b_1,\ldots,b_{n-1})\}, \text{ if } n > 1$$

For example, if f is a (partial) function from M^2 to M_t, i.e. a (many-one) relation between ordered pairs of individuals and truth values, then $dom(f)$ is a set of ordered pairs, i.e. a binary relation, so $dom(dom(f))$ is the domain of *that* relation.

Second, we recall the notion of *relativization* of (total) quantifiers. Expressed in the RT framework, if for every M, Q_M is a relation between $R_1,\ldots,R_k$, where R_i is an n_i-ary relation over M, then the relativized quantifier Q^{rel} has one extra set argument and is defined by

$$Q^{\mathrm{rel}}_M(A,R_1,\ldots,R_k) \iff Q_A(R_1 \cap A^{n_1},\ldots,R_k \cap A^{n_k})$$

This is an important notion for natural language quantifiers, since almost all such quantifiers turn out be relativized, which in turn explains significant facts about the way quantification works in natural language (see Peters and Westerståhl 2006, ch. 4). We may also note that relativized quantifiers are automatically EXT.

Now, using the above, together with Facts 11 and 12, one proves the following result.

Proposition 14
If Q is a quantifier of type $\langle\langle e\ldots e,t\rangle,t\rangle$ in TFT$^+$, then, for all $f \in M^{PFT^+}_{\langle e\ldots e,t\rangle}$,

(a) $(Q^R)^{p1}_M(f) = T \iff \pi^{-1}(Q)_M(f^{+M})$

(b) $(Q^R)^{p2}_M(f) = T \iff \pi^{-1}(Q)^{\mathrm{rel}}_M(dom(dom(f)),f^{+M})$

Moreover, the proposition generalizes to (total) quantifiers of arbitrary type; I leave the working out of this as an exercise.

What this result tells us is that, although extending total quantifiers to corresponding partial ones is possible and even natural, what can be expressed by these partial quantifiers can already be expressed by the total ones, or their relativizations. Intuitively, this seems to be exactly what one should expect. Consider again the partial predicate *prime* (section 2.4). Suppose I am talking about a bunch (a finite set M) of mathematical objects: natural numbers, reals, functions, sets, etc., and I say

(9) Most (things) are prime.

The Rescher quantifier Q^R = *most things* doesn't apply directly to partial predicates, but we can use either of its two partial versions. With $(Q^R)^{p1}$ we get, letting $prime^{+M}$ be the set of prime numbers in M,

(10) Most things in M are prime^{+M}.

This is expressible by the total Rescher quantifier, but perhaps an unlikely interpretation of my words. It is more plausible that $(Q^R)^{p2}$ was used:

(11) Most numbers in M are prime^{+M}.

By Proposition 14 (b), this is expressible with the (total) *most*, the relativization of Q^R.

5.3. Summing up

We have seen that although total quantifiers themselves aren't very natural from a partial perspective, there are perfectly natural partial versions of them. These partial versions exhibit EXT-like properties in predictable ways (Fact 13); there are more options available for 'constancy' than in the total case. However, if relativization is allowed, interpreting sentences with partial predicates and relations can be done already with the total quantifiers, so there seems to be no real (semantic) *need* for the partial versions.

Of course, there are endless ways of construing partial quantifiers that are *not* versions in any straightforward sense of total ones; hopefully the reader has got a glimpse of the variety of options that partiality allows from the discussion above. It remains to be seen if some of these options are actually 'realized' in natural languages. That is, I am not doubting here that partial predicates and relations are used in language; the issue is what kind of partial quantification, if any, is employed.

As noted, one (the only?) proposal in this direction came from Barwise and Cooper (1981). Seen in the present framework, they considered quantifiers q in PFT^+ of type $\langle\langle e,t\rangle,\langle\langle e,t\rangle,t\rangle\rangle$ (or, more simply, $\langle\langle e,t\rangle\langle e,t\rangle,t\rangle$ if we forget about currying) but restricted attention to the special case when each q_M is a partial function from $M^{\mathrm{TFT}^+}_{\langle e,t\rangle}$ to $[M^{\mathrm{TFT}^+}_{\langle e,t\rangle} \longrightarrow \{\mathrm{T},\mathrm{F}\}]$, so the only partial object is q_M itself; all the other objects involved are total.[8]

A main intuition behind this sort of partiality is the alleged lack of truth value of a sentence like

(12) The three boys went to see a movie.

when there aren't exactly three boys in the discourse universe (or some suitably chosen salient universe). These intuitions can certainly be debated. Peters and Westerståhl (2006) argue that (disregarding such intuitions) the only crucial use of partial quantifiers in Barwise and Cooper (1981) occurs in their notion of *strong* quantifiers, which is used to explain the distribution of noun phrases in existential-there sentences.[9] They also argue that it is doubtful that partiality is really called for in that explanation.

Apart from what has been said in this section, and in Barwise and Cooper (1981) and Peters and Westerståhl (2006), I know of no principled discussion of partial quantifiers in natural language. It may be a topic worth exploring further.

6. Discussion

6.1. EXT versus PERM

In stark contrast with EXT, it is obvious how to formulate invariance properties like ISOM and PERM for arbitrary types, even in a partial framework. Any bijection h from M to M' lifts straightforwardly to a bijection h_τ from $M^{\mathrm{TFT}^+}_\tau$ to $M'^{\mathrm{TFT}^+}_\tau$, for any type τ, so u of type τ is ISOM if for any such h,

$$u_{h(M)} = h_\tau(u_M)$$

PERM is the weaker condition which only concerns permutations of M:

$$u_M = h_\tau(u_M)$$

Exactly the same goes for PFT$^+$, since a bijection from M to M' lifts equally straightforwardly to a bijection from $M_\tau^{\mathrm{PFT}^+}$ to $M_\tau'^{\mathrm{PFT}^+}$.

This just re-emphasizes the familiar fact that EXT and PERM (or ISOM) are completely different conditions. Further illustration is afforded by a quick look at which operators have the respective properties in various types. For example, in TFT$^+$, the only PERM operator of type $\langle e,e\rangle$ is the identity function, whereas there are no PERM operators of type $\langle\langle e,t\rangle,e\rangle$ (cf. van Benthem 1989, section 2.1). Similarly in PFT$^+$: an $\langle e,e\rangle$ type operator u is PERM iff for all M and all $a \in M$, if $u_M(a)$ is defined then $u_M(a) = a$; and there are no PERM operators of type $\langle\langle e,t\rangle,e\rangle$ except the null (everywhere undefined) operator.

By contrast, an operator u of type $\langle e,e\rangle$ in PFT$^+$ is p-EXT iff there exists a fixed global partial function $\mathbf{F}$ (like the successor function S) such that for all M and all $a \in M$, if $u_M(a)$ is defined then $u_M(a) = \mathbf{F}(a)$. And there are lots of p-EXT operators of type $\langle\langle e,t\rangle,e\rangle$ in PFT$^+$.

6.2. EXT as a 'constancy' property

This paper has looked at EXT as a reasonable 'constancy' property, worth spelling out for arbitrary types. Two questions could be asked about this strategy. First, are there other similar properties that should also be studied? Second, is EXT really reasonable?

It has been argued that EXT is not quite reasonable, or at least that it is perhaps sufficient for constancy but not necessary, since some familiar quantifiers are not EXT. Notably, the standard universal quantifier $\forall$ (type $\langle\langle e,t\rangle,t\rangle$) is not EXT, but doesn't it 'mean the same' on every universe? The matter is discussed at some length in Peters and Westerståhl (2006) (chapters 3.4 and 4.5), with the tentative conclusion that all natural language quantifiers, *except* some which essentially involve a predicate *thing* that always denotes the universe, are EXT. While this may be a significant observation, it still leaves the issue of a necessary condition for constancy somewhat up in the air.

A slightly different take on the matter might be as follows. In contrast with invariance properties like PERM, EXT is *not closed under definability*. The quantifier $\exists$ is EXT, but not its inner negation (saying of a set that its complement is not empty). Any language with rudimentary means of expression will have the power to refer essentially to the universe, and thereby to define non-EXT operators.[10] In view of this, it would clearly be unreasonable to require that all logical constants are EXT. But a weaker requirement could be that they are all *definable* from (logical and) EXT operators. And this seems indeed to be the case, for all the ‘usual’ logical constants. Moreover, it seems to be an empirical fact that all natural language quantifiers (all quantifiers required in the analysis of natural language) are definable from EXT quantifiers (see Peters and Westerståhl 2006, ch. 4.5). This would be enough to motivate spelling out EXT for arbitrary types, as we have begun to do here.

One may still feel that an analysis of ‘constancy’ has not been accomplished. This leads to the first question asked above: Are there alternative analyses? One such analysis, taking constancy in terms of what is variable and what is constant in valid *inferences*, is sketched in Peters and Westerståhl (2006), ch. 9.3. But if we stick to the idea of constancy as independence (somehow) of the *universe*, there is perhaps one other version worth exploring. The idea, which could be called *rigidity*, would be that a universal operator u is constant over universes if there is a fixed global operator $\mathbf{U}$ such that on each M, u_M is the restriction, in some sense, of $\mathbf{U}$ to M. This idea is clearly very similar to EXT, but not exactly the same. It might even avoid some of the problems we ran into with EXT. Its proper formulation, and the exact relation to EXT, must however be left for another occasion.

6.3. Further issues

If the goal of this paper was to find the proper formulation of EXT for arbitrary types, the result so far is at least incomplete, and at worst a failure. But sometimes there is illumination even in failure. For example, it may be a mistake to look for *the* proper formulation of EXT. Perhaps there are several; we have suggested three, preliminarily called p-$\mathrm{EXT}^{gq,s}$, p-EXT^{gq}, and p-EXT, each corresponding in some way to EXT for total (strictly) relational types. We managed to get an idea of what these amount to for partial quantifiers, and we noted that the last two make good sense for arbitrary types.

However, our findings are certainly incomplete. For one thing, one would like to know if there is a good notion of EXT for total operators of types other than the strictly relational ones, and if so, how it correlates with the formulations for the partial case that we found. For another, whereas EXT is unproblematic for all (strictly) relational types in the total case, this is not so for the partial case. The reason is that many of the facts we observed about the various formulations of partial EXT relied on the assumption that we were dealing with first-order quantifiers, or, more generally, with operators of types whose level is at most 2. Are they generalizable, or does something important happen at level 2? Along with the issues mentioned in the previous subsection, these are questions for further study. Their answers, like the preliminary ones found in this paper, would be small pieces of the puzzling question of what characterizes a logical constant.[11]

Notes

1. Another variant has instead *product types* and unary function types; the difference between that system and TFT^+ is negligible here.
2. For the record, we should make clear exactly what (set-theoretic) object a partial function is. We identify a (unary) *partial function from A to B* with a many-one relation (set of ordered pairs) whose domain is a subset of A and whose range is a subset of B. It is total iff the domain is equal to A. Similarly for partial functions with several arguments. This has the consequence that a partial function from A to B is automatically a partial function from any superset of A to B.
3. It is common but somewhat misleading to call N and B *truth values*. The labels may seem fine but the question is if our notion of truth can really make sense of truth values other than T and F. As Muskens points out, it is better, and in accordance with the picture that PRT presents, to think of T,F,N,B as *truth combinations* rather than truth values: 'true and not false', 'false and not true', 'neither true nor false', and 'both true and false', respectively (thus preserving the intuitive idea of only two truth values). But the common practice is to talk about 3- and 4-*valued* logics.
4. More exactly, he considers PRT equipped with a formal language and a proof system, calling the resulting relational *type theory* TT_2^4, and similarly for TFT_4, with a (total) functional type theory called TY_2^4. The language for TT_2^4 is a sub-language of the language for TY_2^4 (both *languages* are functional, based on lambda abstraction and application), and he shows that a sentence φ in the TT_2^4-language is a TT_2^4 logical consequence of a set of sentences Γ in that language (with respect to models based on the frames for PRT defined above) if and only if φ is a TY_2^4 logical consequence of Γ.
5. Farmer studies a variant of PFT (i.e. PFT^+ with unary functions and currying), letting individual types allow partial functions but not Boolean types, and using a classical (2-valued) logic to describe this system. However, he does not seem to be aware of the problem mentioned earlier with currying partial functions.
6. In general, for quantifier type $\langle \sigma_1 \ldots \sigma_n, t\rangle$ we get the condition:

   If $(f_1,\ldots,f_n) \in dom(q_M)$, $(g_1,\ldots,g_n) \in dom(q_{M'})$, and $f_i^{+M} = g_i^{+M'}$, $1 \le i \le n$, then $q_M(f_1,\ldots,f_n) = q_{M'}(g_1,\ldots,g_n)$.

   Similarly for the other versions of EXT below.
7. We really mean $u_{M'} \restriction M^{\mathrm{PFT}^+}_{\langle \sigma, t\rangle}$, of course, but since any domain in PFT^+ is determined by the set of individuals, the shorter notation $u_{M'} \restriction M$ makes sense. Similarly for the general case of a type $\langle \sigma_1 \ldots \sigma_n, \tau\rangle$. For completeness, we should also stipulate that an operator of type e is never p-EXT, whereas an operator of type t always is.

8. Concerning EXT for such quantifiers, two options seem natural (and nothing in what Barwise and Cooper say indicates which one would be preferable): Whenever $A, B \subseteq M \subseteq M'$ (we simplify matters by using sets rather than characteristic functions as arguments of q):

   (i) If $q_M(A,B)$ and $q_{M'}(A,B)$ are both defined, then $q_M(A,B) = q_{M'}(A,B)$.

   (ii) If $q_M(A,B)$ is defined, then $q_{M'}(A,B)$ is also defined and $q_M(A,B) = q_{M'}(A,B)$.

   Indeed, modulo the trouble of expressing this for characteristic functions instead, (i) is p-EXT^{gq} (cf. footnote 6 and use the formulation (8)), which is equivalent to p-EXT^{gq,s} for these quantifiers, since all arguments are total, and (ii) is p-EXT.
9. Roughly, the problem is to explain why *There are several/at least two/no girls in the garden* is fine, whereas *There are most/the five/all girls in the garden* is not. See Peters and Westerståhl (2006), ch. 6.3, for a detailed overview of this issue.
10. For example, if the language has individual variables (or corresponding expressions) and identity, or 1-place predicates and Boolean connectives, or a predicate *thing*.
11. Work on this paper was supported by a grant from the Swedish Research Council. I would like to thank Reinhard Muskens for some helpful remarks on type systems.

References

Barwise, Jon, and Robin Cooper
1981 Generalized quantifiers and natural language. *Linguistics and Philosophy* 4: 159–219.

Belnap, Nuel
1977 A useful four-valued logic,. In *Modern Uses of Multiple-Valued Logic*, J. M. Dunn, and G. Epstein (eds.), 8–37. Dordrecht: Reidel.

Bonnay, Denis
2006 *Qu'est-ce qu'une Constante Logique?* Paris: Dissertation, IHPST (Université de Paris I).

Farmer, William B.
1990 A partial functions version of church's simple theory of types. *The Journal of Symbolic Logic* 55: 1269–1291.

Feferman, Solomon
1999 Logic, logics, and logicism. *Notre Dame Journal of Formal Logic* 40: 31–54.

Gallin, D.
1975 *Intensional and Higher-Order Modal Logic*. Dordrecht: North-Holland.

Lindström, Per
1966 First-order predicate logic with generalized quantifiers. *Theoria* 32: 186–195.

Montague, Richard
1974 *Formal Philosophy* (edited and with an introduction by Richmond Thomason). New Haven: Yale University Press.

Muskens, Reinhard
1989a *Meaning and Partiality*. Amsterdam: Dissertation, University of Amsterdam.
1989b A relational formulation of the theory of types. *Linguistics and Philosophy* 12: 325–346.

Peters, Stanley, and Dag Westerståhl
2006 *Quantifiers in Language and Logic*. Oxford: Oxford University Press.

van Benthem, Johan
1989 Logical constants across varying types. *Notre Dame Journal of Formal Logic* 30: 315–342.
2002 Logical constants; the variable fortunes of an elusive notion. In *Reflections of the Foundations of Mathematics: Essays in Honor of Solomon Feferman* (Lecture Notes in Logic 15), Wilfrid Sieg et al. (ed.), 420–440. Natick, MA: AK Peters Ltd.

van Benthem, Johan, and Kees Doets
1983 Higher-order logic. In *Handbook of Philosophical Logic, Vol I*, Dov Gabbay, and Franz Guenthner (eds.), 275–329. Dordrecht: D. Reidel.