The traditional square of opposition and generalized quantifiers*

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Abstract

The traditional square of opposition dates back to Aristotle's logic and has been intensely discussed ever since, both in medieval and modern times. It presents certain logical relations, or oppositions, that hold between the four quantifiers *all*, *no*, *not all*, and *some*. Aristotle and traditional logicians, as well as most linguists today, took *all* to have existential import, so that "All As are B" entails that there are As, whereas modern logic drops this assumption. Replacing Aristotle's account of *all* with the modern one (and similarly for *not all*) results in the modern version of the square, and there has been much recent debate about which of these two squares is the 'right' one.

My main point in the present paper is that this question is not, or should not primarily be, about existential import, but rather about patterns of negation. I argue that the modern square, but not the traditional one, presents a general pattern of negation one finds in natural language. To see this clearly, one needs to apply the square not just to the four Aristotelian quantifiers, but to other generalized quantifiers of that type. Any such quantifier spans a modern square, which exhibits that pattern of negation but, very often, not the oppositions found in the traditional square. I provide some technical results and tools, and illustrate with several examples of squares based on quantifiers interpreting various English determiners. The final example introduces a second pattern of negation, which occurs with certain complex quantifiers, and which also is representable in a square.

1 Introduction

A square of opposition illustrates geometrically how certain operators are logically related via certain forms of opposition or negation. In the *quantified square*

*This paper is a concise account of one aspect of my work on quantification and negation over the past years, begun in [10], continued in [8], and presented on various occasions; recently at the 1st Square Conference in Montreux, June 2007, and the 10th Mathematics of Language Workshop at UCLA, July 2007. I thank the audiences at these meetings for helpful questions and remarks. In addition, comments by Larry Horn and Stanley Peters have been valuable to me. Work on the paper was supported by a grant from the Swedish Research Council.

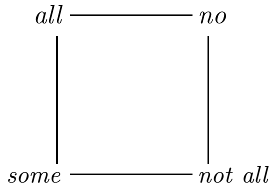

Figure 1: The quantified square

(Figure 1), the operators are the Aristotelian *quantifiers*, but they can also be *modal*, or *temporal*, or *propositional* operators. In these squares we have ordinary, *contradictory* or *outer* negation along the diagonals, and various other forms of negation along the edges. Precisely what these other forms are is a matter of debate, as we will see presently. But an easily identifiable rough pattern is always present, and can be found for other operators as well. For example, the *deontic* square replaces *necessary* by *obligatory*, *impossible* by *prohibited*, and *possible* by *permitted* (Figure 2).

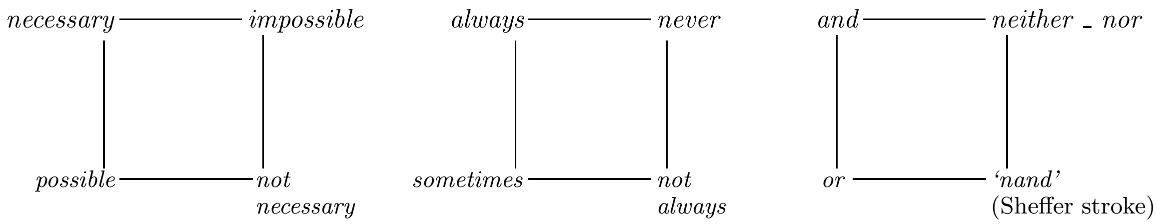

Figure 2: Various squares

The quantified square is basic, in the sense that it reappears in the explanation of the meaning of the operators in the other squares. For example, a conjunction is true iff *all* its conjuncts are true, and a disjunction is true iff *some* of the disjuncts are. Likewise, a proposition is necessary iff it is true under *all* circumstances (possible worlds), possible if true under *some* circumstance, and impossible if true under *none*. In this paper I focus on the quantified square.

2 Aristotle's square

The *Aristotelian square* was described in words by Aristotle, and drawn as a diagram by Apuleios and Boethius some 800 years later (Figure 3). Here the only non-traditional notation used is all_{ei}, which is the universal quantifer with *existential import*, i.e. such that $all_{\mathrm{ei}}(A, B)$ is true iff A is non-empty and included in B. This is what Aristotle and most of his medieval followers, in agreement with many philosophers and most linguists today, took words like

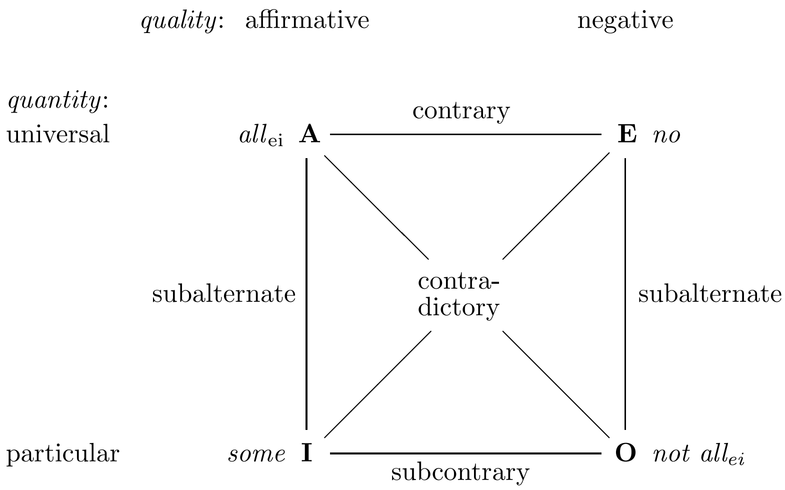

Figure 3: The Aristotelian square

"all" and "every" to mean. In contrast, Frege and most logicians after him take them to stand for the quantifier *all* without existential import: $all(A, B) \Leftrightarrow A \subseteq B$. Now, what certain words in certain languages *mean* looks like an empirical question, one that could be settled by appropriate investigations. Also, the difference between the two suggestions seems rather small. But the thrust of the present paper is that, in the context of the square, this issue is just an indicator of a much more principled one, namely, the nature of negation. Before I get to that, however, let me describe the content of the Aristotelian square.

The (medieval) naming **A**, **E**, **O**, **I** of the four corners is handy and will be used in what follows. As to the logical relations depicted in the square, we have, aside from contradictory negation along the two diagonals, the relations of contrariety, subcontrariety, and subalternation along the sides. These are seen as relations not between the quantifiers themselves but the corresponding statements made with them. Thus, φ and ψ are *contrary* iff they cannot both be true (but can both be false), *subcontrary* iff they cannot both be false (but can both be true), and ψ is *subalternate* to φ iff whenever φ is true, so is ψ.

That the **A** and **E** corners are contrary clearly shows that the universal quantifier has existential import in this square. The same conclusion follows from the fact that **I** is held to be subalternate to **A**. The diagram also contains the classification of statements according to *quality*: they are either *affirmative* or *negative*; negative statements don't make existential claims in the Aristotelian square, but affirmative ones do. And the final classification concerns *quantity*, i.e. whether a statement is *universal* or *particular*.

3 The square as a general pattern of negation

Numerous issues related to the square of opposition have been discussed in the literature, among them the question of existential import at the **A** corner (and possible other corners).[1] My focus in this paper, however, is rather on the square as a pattern of negation in natural languages. The four Aristotelian quantifiers exhibit this pattern, but they are just one instance. Indeed, *any* (generalized) quantifier (of type $\langle 1, 1\rangle$) spans a square, provided the square is taken in the modern (Fregean) way, not in the classical way.

The modern version of the Aristotelian square is drawn in Figure 4. At first

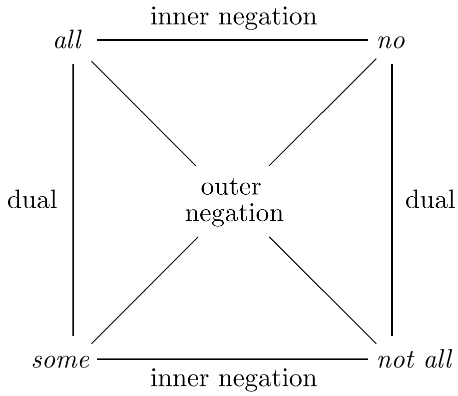

Figure 4: The modern version of Aristotle's square

sight, one may get the impression that the only difference is that *all* has replaced all_{ei} at the **A** and **O** corners (I continue to use these names of the corners, but without writing them in the diagrams). Likewise, if one disregards empty terms, the two squares seem to coincide. As noted, the difference might appear to boil down to different views about existental import. But this impression is misleading, and vanishes as soon as one considers other quantifiers. Instead, the main difference concerns the choice of relations along the sides of the square. Existential import is a side issue in this context.

In the modern square there are only two relations along the sides: inner negation and dual, and like outer negation they can be seen as operations on the quantifiers themselves:

(1) a. The *outer negation* of Q, $\neg Q$, is defined by: $\neg Q(A, B) \Leftrightarrow$ not $Q(A, B)$

b. The *inner negation* of Q, $Q\neg$, is defined by: $Q\neg(A, B) \Leftrightarrow Q(A, M - B)$ (where M is the universe)

[1] Horn [3] is a classic history of negation, with thorough discussion of many related issues. Pragmatic views on existential import are defended in Horn [4] and Peters and Westerståhl [8], ch. 4.2.1.

c. The *dual* of Q, Q^{d}, is defined by: $Q^{\mathrm{d}} = \neg(Q\neg) = (\neg Q)\neg$

So in Figure 4 we have the same relation along both horizontal sides, in contrast with the Aristotelian square. More importantly, this relation has little to do with (sub)contrariety. In general, as we will see, nothing prevents Q and $Q\neg$ from being both true, or both false, of the same arguments. Likewise, the relation along the vertical sides has little or nothing to do with subalternation, since we may easily have Q true and Q^{d} false of the same arguments.

To appreciate these points, we need the general notion of a quantifier.

4 Quantifiers

Only the bare definitions follow; for (much) more about quantifiers, see [8].

(2) A (generalized) *quantifier Q of type* $\langle 1,1\rangle$ associates with each universe M a binary relation Q_M between subsets of M.

Many such quantifiers interpret English simple or complex determiners, and we can conveniently name them accordingly, as in the following examples. ($|X|$ is the cardinality of the set X). For all M and all $A, B \subseteq M$,

$\mathit{all}_M(A,B) \iff A \subseteq B$

$(\mathit{all}_{\mathrm{ei}})_M(A,B) \iff \emptyset \neq A \subseteq B$

$\mathit{no}_M(A,B) \iff A \cap B = \emptyset$

$\textit{at least two}_M(A,B) \iff |A \cap B| \geq 2$

$\textit{exactly five}_M(A,B) \iff |A \cap B| = 5$

$\textit{all but three}_M(A,B) \iff |A - B| = 3$

$\textit{more than two-thirds of the}_M(A,B) \iff |A \cap B| > 2/3 \cdot |A|$

most = *more than half of the*

$\textit{John's}_M(A,B) \iff \emptyset \neq A \cap \{a : \text{John `possesses' } a\} \subseteq B$

$\textit{no \_ except John}_M(A,B) \iff A \cap B = \{\text{John}\}$

$\textit{infinitely many}_M(A,B) \iff A \cap B$ is infinite

$\textit{an even number of}_M(A,B) \iff |A \cap B|$ is even

Also:

Let **1** (**0**) be the trivially true (false) quantifier (e.g. **1** = *at least zero*, **0** = *fewer than zero*).

The following properties of quantifiers will be relevant:

(3) a. Q is *conservative* (CONSERV) iff $Q_M(A,B) \Leftrightarrow Q_M(A, A\cap B)$

b. Q satisfies *extension* (EXT) iff for $A, B \subseteq M \subseteq M'$, $Q_M(A,B) \Leftrightarrow Q_{M'}(A,B)$

c. Q is *closed under isomorphism* (Isom) iff for $A, B \subseteq M$ and $A', B' \subseteq M'$, $|A-B| = |A'-B'|$, $|A \cap B| = |A' \cap B'|$, $|B-A| = |B'-A'|$, and $|M-(A \cup B)| = |M'-(A' \cup B')|$, entails $Q_M(A,B) \Leftrightarrow Q_{M'}(A',B')$

Conserv and Ext together mean that quantification is in effect restricted to the first argument ([8], ch. 4.5). All quantifiers interpreting natural language determiners satisfy these two properties. Many satisfy Isom as well; in the list above, all do except those mentioning particular individuals, i.e. all except *John's* and *no _ except John*.

Ext entails that the universe is irrelevant, so we may drop the subscript $_M$ for such quantifiers, as we in effect did in the definition (1) of the various negations. Here is a first manifestation of the difference between the negations in the modern square and the classical oppositions:

Fact 1
The combination Conserv + Ext *is preserved under inner and outer negation (and hence under duals), but (sub)contrariety is not. I.e. there are (sub)contrary quantifiers Q and Q' such that Q but not Q' satisfies* Conserv *and* Ext.

Note also that Conserv + Ext entails that the definition of inner negation may be expressed as follows:

(4) $\quad Q\neg(A,B) \iff Q(A, A-B)$

Below I assume that these two properties hold of all quantifiers mentioned.

5 Modern vs. classical squares

Every type $\langle 1,1 \rangle$ quantifier *spans* a (modern) square. Define:

(5) $\quad \mathit{square}(Q) = \{Q, \neg Q, Q\neg, Q^{\mathrm{d}}\}$

Fact 2
(a) $\mathit{square}(\mathbf{0}) = \mathit{square}(\mathbf{1}) = \{\mathbf{0}, \mathbf{1}\}$.
(b) *If Q is non-trivial, so are the other quantifiers in its square.*
(c) *If $Q' \in$ square(Q), then square(Q) = square(Q').*
(d) *square(Q) has either two or four members.*

By (c), any two squares are either identical or disjoint. As to (d), a square normally has four members, but it can happen that $Q = Q\neg$ (and thus $Q^{\mathrm{d}} = \neg Q$), and then it has two.

Example 3
$|\mathit{square}(Q)| = 2$ when Q expresses identical conditions on $A \cap B$ and $A - B$ (cf. (4) above), such as the quantifier

$$\mathit{exactly\ half}(A,B) \iff |A \cap B| = |A - B|$$

A more spectacular example is due to Keenan [5], who noted that the sentences

(6) a. Between one-third and two-thirds of the students passed.
b. Between one-third and two-thirds of the students didn't pass.

are logically equivalent, i.e. that if $Q =$ *between one-third and two-thirds of the*, then $Q = Q\neg$.

Note that $square(all_{\mathrm{ei}})$ is *not* the Aristotelian square: besides all_{ei} and its outer negation, it contains no_{ei} and its outer negation, which holds of A, B iff either A is empty or $A \cap B$ is non-empty. This may seem like a rather unnatural square; certainly no one would take the last-mentioned quantifier to interpret the word "some". In my opinion, the unnaturalness is a consequence of taking "all" (or "no") to have existential import, but that is not a main point here. The main point is that using a classical notion of square instead might save the Aristotelian square, but not any others. That is, the pattern in the Aristotelian square doesn't generalize, and therefore isn't a common pattern of negation. To make this precise, let us introduce the following notion.

(7) A *classical square* is an arrangement of four quantifiers as traditionally ordered and with the same logical relations – contradictories, contraries, subcontraries, and subalternates – between the respective positions.

Now each position determines the diagonally opposed quantifier, i.e. its outer negation, but *not* the quantifiers at the other two positions. For example:

Fact 4
The square

[**A**: *at least five*; **E**: *no*; **I**: *some*; **O**: *at most four*]

is classical. More generally, for $n \geq k$,

[**A**: *at least n*; **E**: *fewer than k*; **I**: *at least k*; **O**: *fewer than n*]

is classical.

The classical squares in Fact 4 look very unnatural. There is no interesting sense, it seems, in which *no* is a negation of *at least five* or *at most four*. So it is no accident that the debate surrounding the square to a large extent has been restricted to the four Aristotelian quantifiers. In contrast, the modern square exhibits a completely general pattern of negation for type $\langle 1, 1\rangle$ quantifiers, and thereby for expressions denoting such quantifiers; expressions that form a rich and productive class in many languages.[2]

[2] The observations here about negation and opposition collect facts that are essentially known from the literature; e.g. Barwise and Cooper [1], Keeenan and Stavi [6], Brown [2], Loebner [7]. But the behavior of modern squares of opposition in the context of natural language has not, to my knowledge, been systematically studied as is done in this paper.

6 The A, E, O, and I corners

While *square*(Q) uniquely specifies the *quantifiers* involved, it says nothing about how to distinguish the *corners*. Can we also find quantitative and qualitative aspects in the squares? Is this possible in an arbitrary quantified square?

In general, the answer is No. But in many cases we can obtain an identification, or at least a partial one, by suitably generalizing features of those corners in the Aristotelian square. I will look at three ways of doing this.

Monotonicity

A striking feature of the quantifiers in the Aristotelian square is their monotonicity behavior. Indeed, these quantifiers are *doubly monotone* (with a small *caveat* for the **A** and **O** corners).

(8) a. Q is *right monotone increasing* ($\textsc{mon}\!\uparrow$) iff
$Q(A, B)$ & $B \subseteq B' \Rightarrow Q(A, B')$

b. Q is *right monotone decreasing* ($\textsc{mon}\!\downarrow$) iff
$Q(A, B)$ & $B' \subseteq B \Rightarrow Q(A, B')$

c. Q is *left monotone increasing* ($\uparrow\!\textsc{mon}$) iff
$Q(A, B)$ & $A \subseteq A' \Rightarrow Q(A', B)$

d. Q is *left monotone decreasing* ($\downarrow\!\textsc{mon}$) iff
$Q(A, B)$ & $A' \subseteq A \Rightarrow Q(A', B)$

Q is doubly monotone if it has both a left and a right monotonicity property. For example, *all* is $\downarrow\!\textsc{mon}\!\uparrow$. all_{ei} is $\textsc{mon}\!\uparrow$ but only *weakly* $\downarrow\!\textsc{mon}$, in the sense that $Q(A, B)$ & $\emptyset \neq A' \subseteq A \Rightarrow Q(A', B)$ (cf. also section 10 below).

Fact 5

([8], ch. 5) *The monotonicity behavior of Q determines that of all elements of square(Q):*

(a) *Q is* $\textsc{Mon}\!\uparrow$ *iff $Q\neg$ is* $\textsc{Mon}\!\downarrow$ *iff $\neg Q$ is* $\textsc{Mon}\!\downarrow$ *iff Q^d is* $\textsc{Mon}\!\uparrow$

(b) *Q is* $\uparrow\!\textsc{Mon}$ *iff $Q\neg$ is* $\uparrow\!\textsc{Mon}$ *iff $\neg Q$ is* $\downarrow\!\textsc{Mon}$ *iff Q^d is* $\downarrow\!\textsc{Mon}$

So if Q is *doubly* monotone, all four combinations are exemplified in its square. This means that right monotonicity could be seen as *quality*, with $\textsc{Mon}\!\uparrow$ as *affirmative* and $\textsc{Mon}\!\downarrow$ as *negative*. And left monotonicity could be seen as *quantity*, with $\uparrow\!\textsc{Mon}$ as *particular* and $\downarrow\!\textsc{Mon}$ as *universal*. Thus, we can identify the exact position in the square of any doubly monotone quantifier.

However, many quantifiers are only *right monotone*, like the proportional quantifiers. So we can say, for example, that *at least two-thirds of the* is affirmative: it is either **A** or **I**, but we cannot say which. In one sense, this is not unreasonable: the dual of *at least two-thirds of the* is *more than one-third of the*, and it may seem arbitrary which of these two should go into the **A** position. As we will see, however, our third criterion suggests that *at least two-thirds of the* belongs in the **A** position.

Symmetry
What can we say about quantifiers that are neither left nor right monotone, such as *an even number of* or *exactly ten*? Some help may come from a property identified and discussed already by Aristotle, who noted that the *order* between the two arguments of the quantifier is irrelevant at the **I** and **E** corners.

(9) a. Q is *symmetric* iff $Q(A, B) \Rightarrow Q(B, A)$.

b. Q is *co-symmetric* iff $Q\neg$ is symmetric.

Fact 6
([8], ch. 6.1) *The symmetry behavior of Q determines that of all elements of square(Q): Q is symmetric iff $Q\neg$ is co-symmetric iff $\neg Q$ symmetric iff Q^d is co-symmetric.*

Thus, when a quantifier has symmetry behavior, Fact 6 allows us to distinguish the **I** and **E** from the **A** and **O** corners. In particular, if Q is *right monotone* and *either symmetric or co-symmetric*, we can again pinpoint its exact position in the square, given that the **I** and **E** positions are symmetric, and the **A** and **O** positions co-symmetric. For example, *at most ten* is at the **E** position. But we already knew that, since *at most ten* is $\downarrow$Mon$\downarrow$. Indeed, if Q is right monotone and symmetric, it is clearly also left monotone. In fact, the cases where symmetry would give extra information are somewhat limited. This is also illustrated by the next result. Let Fin mean that attention is restricted to finite universes.

Fact 7
(Isom, Fin) *If Q is* Mon$\uparrow$ *and symmetric, then $Q =$ at least n, for some $n \geq 0$.*

So we only get extra information for cases like *an even number of*, which satisfies all assumptions of Fact 7 except monotonicity, and *no _ except John*, which is symmetric but not Isom or right monotone. This gives us two possible configurations of *square*(*an even number of*) and *square*(*no _ except John*).

We should also ask, however, if the two criteria for positioning quantifiers in squares, monotonicity and symmetry, can ever conflict with each other. After all, the intuitions behind them are rather different. A conflict would occur if we found a symmetric quantifier that was also either $\downarrow$Mon$\uparrow$ or $\uparrow$Mon$\downarrow$ (and correspondingly for co-symmetry). Fortunately, this cannot happen:

Fact 8
If Q is symmetric and either $\downarrow$Mon$\uparrow$ *or* $\uparrow$Mon$\downarrow$*, Q is trivial* (*either* **0** *or* **1**).

Thus, the intuitions behind identifying the corners by means of monotonicity and symmetry are quite robust.

Just as the selection of monotone and symmetric quantifiers is rather restricted (Fact 7), so is the choice of doubly monotone quantifiers, at least when Isom is presupposed. One can show the following (cf. [10], sect. 4.3):

Theorem 9 (ISOM, FIN) $\uparrow$MON$\uparrow$ *quantifiers are finite disjunctions of quantifiers of the form* at least n of the k or more, *i.e.* $|A| \geq k$ & $|A \cap B| \geq n$ $(0 \leq n \leq k)$. *More generally,* $\uparrow$MON *quantifiers are finite disjunctions of quantifiers of the form* $|A \cap B| \geq n$ & $|A - B| \geq k$.

However, there is an interesting class of non-ISOM quantifiers with significant monotonicity properties: the possessives. For example, *at least five of John's* is $\uparrow$MON$\uparrow$, hence belongs to the **I** corner. And (*all of*) *John's* is MON$\uparrow$ and *weakly* $\downarrow$MON (you can decrease A as long as something belonging to John remains), so it goes in the **A** corner. On the other hand, *most of John's* is MON$\uparrow$ but not left monotone, so at least we know it is affirmative (**A** or **I**).[3]

Reduction
In addition to monotonicity and symmetry, one can sometimes use a more pragmatic and loose criterion: *square*(Q) should reduce to the (modern) Aristotelian square in *limiting cases*. What a limiting case is varies with the type of quantifier considered, but in many examples it is rather clear. For example, to return to the proportional quantifiers discussed above, it is natural to take $p = q$ as the limiting case of *at least p/q of the*, and then $|A \cap B| \geq p/q \cdot |A|$ reduces to $|A \cap B| \geq |A|$, i.e. $A \subseteq B$ (assuming FIN, which is reasonable for proportional quantifiers). This motivates placing *at least p/q of the* in the **A** corner, and its dual *more than (q-p)/q of the* in the **I** corner (see section 8 below).

So in many cases, it is in fact possible to identify the **A**, **E**, **O**, and **I** corners of a given square, as will be illustrated by the examples to follow in the remainder of the paper. Note that, by Fact 2, it is enough to identify *one* corner of the square; then the others are fixed too.

7 Numerical quantifiers

Let a *numerical* quantifier be one of the form *at least n* ($n \geq 0$), or a Boolean combination (including inner negation) of such quantifiers: *at most n*, *more than n*, *all but n*, *exactly n*, *between k and n*, etc. Figure 5 presents *square*(*at least six*) and Figure 6 *square*(*exactly five*).[4] The quantifiers in Figure 5 are doubly monotone, so there is no question about the identification of the appropriate corners. In the limiting case, when *six* is replaced by *one* (or *five* by *zero*) we get *square*(*all*). In Figure 6 there is no monotonicity, but we have symmetry and co-symmetry, so *exactly five* should be either at the **E** or the **I** corner. The choice made in Figure 6 to place it at the **E** corner is dictated by the third criterion for identifying the corners: with *five* replaced by *zero*, we again obtain *square*(*all*).

[3]See [8], ch.7.12, for these and other monotonicity facts about possessives.

[4]Here and below I use italics to indicate a quantifier interpreting a corresponding English determiner, so that one can see directly which corners of the square are 'realizable' as simple or complex determiners. Also, the corners are always understood to be oriented as in the Aristotelian square.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{r} \textit{all but at most five } A \text{ are } B \\ |A - B| \leq 5 \end{array} \arrow[r, dash] \arrow[d, dash] & \begin{array}{l} \textit{at most five } A \text{ are } B \\ |A \cap B| \leq 5 \end{array} \arrow[d, dash] \\
\begin{array}{r} \textit{at least six } A \text{ are } B \\ |A \cap B| \geq 6 \end{array} \arrow[r, dash] & \begin{array}{l} \text{``all but at least six } A \text{ are } B\text{''} \\ |A - B| \geq 6 \end{array}
\end{tikzcd}
$$

Figure 5: *square*(*at least six*)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{r} \textit{all but five } A \text{ are } B \\ |A - B| = 5 \end{array} \arrow[r, dash] \arrow[d, dash] & \begin{array}{l} (\textit{exactly}) \textit{ five } A \text{ are } B \\ |A \cap B| = 5 \end{array} \arrow[d, dash] \\
\begin{array}{r} \text{``not five } A \text{ are } B\text{''} \\ |A \cap B| \neq 5 \end{array} \arrow[r, dash] & \begin{array}{l} \text{all but five } A \text{ are not } B \\ \text{(with wide scope of ``not'')} \\ |A - B| \neq 5 \end{array}
\end{tikzcd}
$$

Figure 6: *square*(*exactly five*)

These squares are perhaps not very exciting, but there is nothing wrong with them. The truth conditions at each corner are clear, and one sees how English 'realizes' at least five of the eight corners by means of determiners. No determiners seem to correspond to the **I** and **O** corners of *square*(*exactly five*).

But the point is that these squares are not classical. For example, $|A \cap B| \leq 5$ and $|A - B| \leq 5$ are compatible (provided $|A| \leq 10$), so they are not contraries. And $|A \cap B| = 5$ does not imply $|A - B| \neq 5$ (unless $|A| \neq 10$). Will the squares become classical under suitable presuppositions, just as *square*(*all*) becomes classical if one presupposes that A is non-empty? They will, but the required presuppositions are unreasonable. This is seen from the next fact.

Fact 10
(a) *square(*at least $n+1$*) is classical iff* $|A| > 2n$ *is presupposed.*
(b) *square(*exactly n*) is classical iff* $|A| \neq 2n$ *is presupposed.*

Obviously, it makes no sense at all to have

(10) Five students passed the exam.

presuppose that the number of (salient) students was distinct from ten. *Exactly five* simply doesn't fit in a classical square of opposition.

8 Proportional quantifiers

at least 2/3 of the A are B $\lvert A \cap B\rvert \geq 2/3 \cdot \lvert A\rvert$	*at most 1/3 of the* A are B $\lvert A \cap B\rvert \leq 1/3 \cdot \lvert A\rvert$
more than 1/3 of the A are B $\lvert A \cap B\rvert > 1/3 \cdot \lvert A\rvert$	*fewer than 2/3 of the* A are B $\lvert A \cap B\rvert < 2/3 \cdot \lvert A\rvert$

Figure 7: *square*(*at least 2/3 of the*)

Q = *at least 2/3 of the* is $\mathrm{MON}\uparrow$, but not left monotone or symmetric, so two configurations are possible: either Q or Q^{d} goes in the **A** corner. But we already saw how our third criterion suggests the first alternative, as in Figure 7.

This time, all four corners are 'realized' as English determiners. And again, the square is not classical. However, in this case it may be possible to argue that there is existential import at each corner, supplied by the definite article *the*. But if $A \neq \emptyset$ is added to the truth conditions, we no longer have a modern square. (though we do have a classical one). This could be taken to favor a presuppositional analysis of proportional quantifiers as regards existential import.

Alternatively, one might consider $square(\textit{at least 2/3 of the}^+)$, where $\textit{at least 2/3 of the}^+(A, B) \Leftrightarrow |A \cap B| \geq 2/3 \cdot |A| > 0$. In this square (which incidentally is both classical and modern), we get the same problem at the **O** and **I** corners that the Aristotelian square had at the **O** corner, i.e. that the truth condition is a disjunction, one of whose disjuncts is $A = \emptyset$. The matter hinges on how one should understand proportional determiners in negation contexts, something that might be worth investigating further, though I will not attempt that here.

9 Exceptive quantifiers

Exceptive quantifiers, i.e. quantifiers involved in the interpretation of exceptive noun phrases, like "every professor except Mary" or "No students except foreign exchange students", have been studied extensively in the literature, (see [8], ch. 8, for an overview of the issues involved and a proposed general analysis). Their interaction with negation is not without interest. Here I just exemplify with the very simplest case (Figure 8).

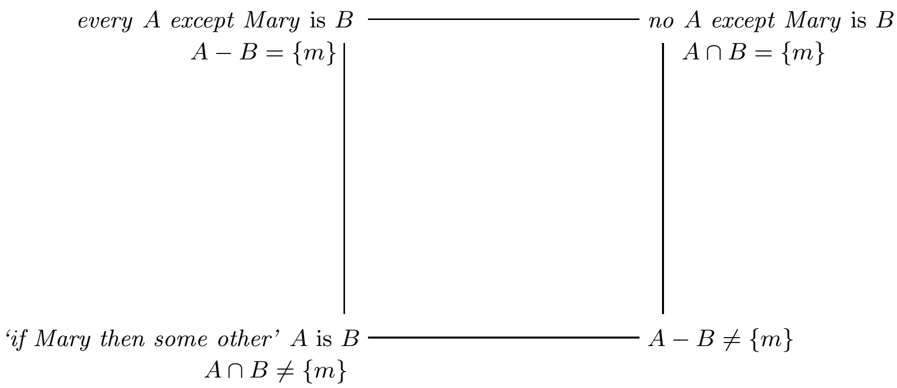

Figure 8: $square(\textit{every \_ except Mary})$

In this square there is (co-)symmetry and no monotonicity. But in the limiting case when the exception set is *empty* we obtain $square(\textit{all})$. So quantifiers of the form $Q(A, B) \Leftrightarrow A - B = \{m_1, \ldots, m_k\}$ $(k \geq 0)$ belong in the **A** corner. $square(\textit{every \_ except John})$ is both modern and classical. The **O** corner appears to be unrealized. A suggestion (from [8], ch. 4.3) has been made in Figure 8 for the **I** corner; it should be taken as *possible* English determiner with the desired interpretation.

10 Possessive quantifiers

The final examples come from possessive constructions. Such constructions generate a rich and interesting class of quantifiers related to natural languages. An account of these quantifiers can be found in [8], ch. 7. [9] develops the account further, including a study of possessives and negation. Here I will only present one of the simplest examples. But already this example indicates some new possibilities for negation to apply in complex quantificational contexts.

We took a sentence like

(11) Mary's pupils are bright.

to mean that Mary 'possesses' at least one pupil and that all of her pupils are bright. In other words, the truth conditions are

(12) $\mathit{Mary's}(A, B) \iff \emptyset \neq A \cap R_m \subseteq B$

where m is Mary, R is the 'possessor relation', and $R_m = \{b : R(m, b)\}$.

Now form *square*(*Mary's*) as usual (Figure 9). By (12), *Mary's* is $\text{M}\small{\text{ON}}\!\uparrow$ and

$$\begin{tikzcd}
\begin{array}{c}\textit{Mary's } A \text{ are } B \\ \emptyset \neq A \cap R_m \subseteq B\end{array} \arrow[r, dash] \arrow[d, dash] & \begin{array}{c}\textit{none of Mary's } A \text{ are } B \\ \emptyset \neq A \cap R_m \subseteq \overline{B}\end{array} \arrow[d, dash] \\
\begin{array}{c}\text{Mary has no } A \text{ or} \\ \text{some of her } A \text{ are } B \\ A \cap R_m = \emptyset \vee A \cap R_m \not\subseteq \overline{B}\end{array} \arrow[r, dash] & \begin{array}{c}\text{Mary has no } A \text{ or} \\ \text{not all of her } A \text{ are } B \\ A \cap R_m = \emptyset \vee A \cap R_m \not\subseteq B\end{array}
\end{tikzcd}$$

Figure 9: *square*(*Mary's*)

weakly $\downarrow$MON (you can decrease A as long as $A \cap R_m \neq \emptyset$), so it belongs in the **A** corner. While in this case the **E** corner provides a natural kind of negation of "Mary's As are B", the **O** corner does not:

(13) Mary's pupils aren't bright.

can hardly mean that *either* Mary has no pupils *or* some of her pupils are not bright. Still, (13) is ambiguous. It can mean what the inner negation yields, that she has pupils but none of them are bright. But it also seems possible to use (13) to express that she has pupils and some of them aren't bright. This becomes clearer if we start instead from a version of the positive statement where the universal quantification is explicit:

(14) All of Mary's pupils are bright.

Now it is rather clear that

(15) All of Mary's pupils aren't bright.

has both readings, whereas

(16) Not all of Mary's pupils are bright.

seems to have only the second reading. But none of these negative statements expresses truth conditions that are found in *square*(*Mary's*)!

It can be shown that the right way to deal with this situation is *not* the strategy hinted at for proportional quantifiers earlier, i.e. to presuppose 'possessive' existential import, so that at all corners in Figure 9, $A \cap R_m \neq \emptyset$ is assumed. Instead, the behavior of sentences like (11) and (14) under negation reflects the fact that these, and in fact all, possessives involve a second quantification over the 'possessions', which is sometimes implicit, sometimes explicit. Moreover, it need not be universal quantification:

(17) Most of Mary's pupils are bright.

In general, possessives involve *two* quantifiers, one over the 'possessors' and one over the 'possessions'.[5] But then there are many more possibilities to apply outer or inner negation, or dual. In particular, there is an operation I will call *middle negation*, obtained by applying (outer) negation to the 'possessions' quantifier, say, Q_2. That is:

(18) $\neg^m(Q_2 \text{ of Mary's}) \;=\; \neg Q_2 \text{ of Mary's}$

There is a corresponding 'middle dual':

(19) $(Q_2 \text{ of Mary's})^{\mathrm{d}^m} \;=\; (Q_2)^{\mathrm{d}} \text{ of Mary's}$

Here I have illustrated with *Mary's*, or rather Q_2 *of Mary's*, but middle negation and dual can be defined for arbitrary possessive quantifiers. Moreover, they generate a new 'square of middle opposition':

(20) $\mathit{square}^m(Q) \;=\; \{Q, Q\neg, \neg^m Q, Q^{\mathrm{d}^m}\}$

This square has the same crucial property as the standard one:

Fact 11
If Q' belongs to square$^m(Q)$, then square$^m(Q') =$ square$^m(Q)$.

For *(all of) Mary's*, we can now check that the relations in the diagram of Figure 10 obtain. Here all four corners are realized as possessive determiners,

[5] For *Mary's*, the 'possessor' is Mary herself, and *Mary* can indeed be taken as a quantifier, holding of a set B just in case B contains Mary. But the paradigmatic case is rather something like "at least two of most teacher's pupils", where it is clear that *most* quantifies over the 'possessors' and *at least two* over the 'possessions'.

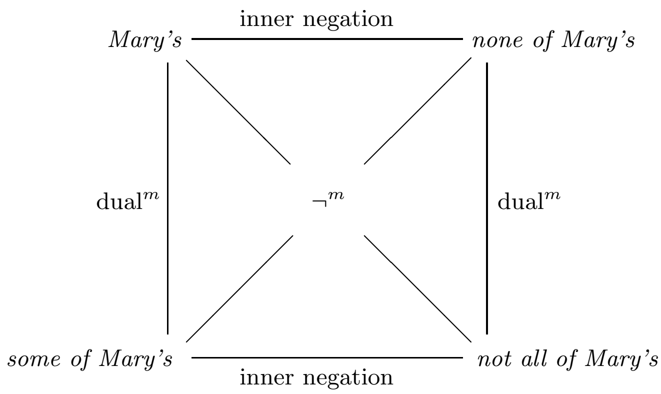

Figure 10: $square^m$(*Mary's*)

and we have in one diagram the two natural ways to negate (11). Further, the semantics itself gives all corners 'possessive' existential import, and we have double monotonicity of the expected kind at each corner.

Much more can be said. First, one can show that of the sixteen possible ways of applying (or not) inner and outer negations and duals to the quantifiers over 'possessors' and 'possessives', respectively, only eight yield distinct results. Second, these eight quantifiers are naturally represented in a single *cube of opposition*. But a detailed exploration of the properties of this cube must be left for another occasion.

11 In conclusion

The square of opposition is a useful conceptual tool for understanding how negation interacts with quantification (and thereby several other operators). It doesn't represent a particular quadruple of quantifiers, such as the four Aristotelian ones, but rather a pattern that recurs for *all* quantifiers. This pattern, I have argued, uses the modern square, with its two basic forms of negation, and their combination with each other, the dual.

I made a few general observations, and used examples to illustrate the variety of negation-quantification interaction in natural language. In particular, complex quantifiers built from simpler ones, such as those one finds with the possessives, indicate even further varieties of such interaction, representable by other squares, or cubes, of opposition.

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