On the Compositional Extension Problem*

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Abstract

A semantics may be compositional and yet *partial*, in the sense that not all well-formed expressions are assigned meanings by it. Examples come from both natural and formal languages. When can such a semantics be extended to a *total* one, preserving compositionality? This sort of *extension problem* was formulated in Hodges [9], and solved there in a particular case, in which the total extension respects a precise version of the fregean dictum that the meaning of an expression is the contribution it makes to the meanings of complex phrases of which it is a part. Hodges' result presupposes the so-called *Husserl property*, which says roughly that synonymous expressions must have the same category. Here I solve a different version of the compositional extension problem, corresponding to another type of linguistic situation in which we only have a partial semantics, and without assuming the Husserl property. I also briefly compare Hodges' framework for grammars in terms of partial algebras with more familiar ones, going back to Montague, which use many-sorted algebras instead.

Keywords: compositional extension, compositionality, fregean extension, Husserl property, partial algebra, term algebra

1 Introduction

The *compositional extension problem* considered in this paper is the following: Suppose we have a compositional semantics (meaning assignment) for a given set of expressions, which is *partial* in the sense that not all well-formed expressions receive meanings. Can we extend this semantics to a total one which is (non-trivial and) still compositional?

This problem arises in the study of semantics for formal as well as natural languages. For example, suppose a linguist's knowledge of a natural language

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she is studying is partial, but that she has been able to assign meanings in a compositional fashion to a significant fragment of it. It is natural to inquire under which conditions the semantics can be extended to the whole language, preserving the already assigned meanings as well as compositionality.

Or, suppose a particular linguistic theory only assigns meanings to expressions of a certain category – say, to sentences but not to non-sentential parts of sentences. This could happen because only sentences are deemed worthy of 'meanings' in some philosophically preferred sense. Or simply because the goal of the theory is to yield truth conditions for sentences. An example from logic is Hintikka's game-theoretic semantics for predicate logic (cf. Hintikka and Sandu [6]), which assigns meanings to sentences but not to formulas with free variables. Still, the partial meaning assignment to sentences is compositional (in a suitable sense), and it is again natural to ask if it can be extended to all the well-formed expressions of the language while preserving compositionality.

The issue of the compositionality of game-theoretic semantics was the starting point of a series of recent papers by Wilfrid Hodges (see [7], [8], and [1], among others), but the framework he used and some of his results reach far beyond that particular example.[1] In [9], he introduced the extension problem mentioned above in a general setting, and proved a significant result – Hodges' Extension Theorem – about compositional extensions. In the present paper I consider generalizations and variants of Hodges' results.

Compositional extension problems show up in other forms too. One case, also discussed by Hodges, is when new words or expressions are added to the language. Again we want to know if our semantics extends in a compositional way. Another case is that of idioms – say, when an already existing phrase acquires an idiomatic meaning. One has to consider how both syntax and semantics should be extended, and whether compositionality can be preserved.[2] Here, however, I only consider the 'pure' extension problem, where there is no change in the syntax, only the question of extending a partial meaning assignment to a total one.

This problem can be formulated in purely algebraic terms, roughly as the

[1] Hintikka had claimed that game-theoretic semantics for his 'independence-friendly' extension of predicate logic (IF logic) is non-compositional. In the process of evaluating that claim, Hodges not only gave compositional versions of game-theoretic semantics but also uncovered interesting aspects of the notion of compositionality, such as the partial vs. total issue that leads to the compositional extension problem, and the so-called 'Husserl Property', to be discussed below. The debate concerning the compositionality of game-theoretic semantics now seems to be over; cf. Sandu and Hintikka [16]. As a partial semantics, it *is* compositional in the 'core' sense of compositionality (Section 3.7 below), and by Hodges' Extension Theorem (or by a direct construction), it can be extended to a total compositional semantics. But if further requirements are made, in particular the requirement that the 'meaning' of a formula should be a set of assignments to individual variables (as in standard Tarski sermantics for predicate logic), then game-theoretic semantics for IF logic is provably *not* compositional in this stronger sense (this is shown in Cameron and Hodges [1]).

[2] One could say that much of the vast literature about idioms is about ways to solve this problem – despite the fact that (some) particular idioms are often described as 'non-compositional' (see, for example, Nunberg, Sag, and Wasow [15]). In Westerståhl [18] I discuss why this is so, and state the issue explicitly as a compositional extension problem within the present framework.

question of the existence of certain extensions of partial congruence relations on term algebras. The *terms* in these algebras represent analyzed linguistic expressions. In [9], Hodges gives a solution under two extra conditions on the initially given semantics, besides compositionality.

The first condition is *cofinality*, in the sense that any term which is not yet meaningful under the given semantics is a subterm of some already meaningful term. The game-theoretic semantics mentioned above is a case in point, since any formula is a subformula of some sentence. The second condition is called the *Husserl property*; it says roughly that synonymous terms must have the same semantic category. This is an interesting, though not self-evident, property. Under these conditions Hodges shows not only that a total compositional extension always exists, but that there is a unique one with a much stronger property, a so-called *fregean extension*.

I consider the compositional extension problem under weaker assumptions. What happens if we drop cofinality, or the Husserl property, or both? Without any assumptions at all except compositionality, there need not exist any total compositional extension (Example 8, Section 3.8). But (Theorem 12, Section 6) if we replace cofinality and the Husserl property by the – rather modest – assumption that the set of meaningful terms is closed under subterms, then a total compositional extension always exists.

Theorem 12, which is the main result here, is a variation rather than a generalization of Hodges' Theorem, since cofinality is orthogonal to the assumption of closure under subterms: If both cofinality and closure under subterms hold, the given semantics must already be total, so the extension problem becomes trivial. One may also consider retaining the Husserl property but dropping cofinality, or, vice versa, keeping cofinality without the Husserl property. In the first case, the extension problem turns out to have an easy solution (Corollary 11, Section 5), but in the second case it seems to be open.

The particular interest of Hodges' Theorem stems (apart from applications to game-theoretic semantics) from the fact that the notion of a fregean extension explicates the natural idea that

(F) the meaning of a term is the contribution it makes to the meanings of complex terms of which it is a constituent,

which in turn can be taken as one version of Frege's famous Context Principle (say, the 'Contribution Principle'). The assumption of cofinality is natural here, for then each 'new' term is part of an already meaningful term, and has to 'contribute' to the given meanings in the right way.

In other circumstances, however, the assumption of closure under subterms – and hence *not* cofinality – is the natural one, a prime example being when one wants to extend the semantics for a given *fragment* to the whole language. Theorem 12 only shows the existence of a total compositional extension in this case, though under very weak assumptions (notably, without assuming the Husserl property). Such extensions, when they exist, are far from unique (Section 9.1). Yet the argument needed to prove the existence claim turns out to be somewhat subtle.

At this point I would like to dispel a possible misunderstanding. It is sometimes claimed that compositionality is a trivial or empty requirement, since 'any semantics can be made compositional by suitable adjustments'. (Zadrozny [19] is a strong statement of this view; see Westerståhl [17] for a discussion.) But what this usually turns out to mean is that for any meaning assignment to some set of expressions there is *another* meaning assignment to these expressions which is compositional and such that the original meanings are *recoverable* from the new ones. And that claim is indeed trivial. It suffices to take as the new meaning of e the pair consisting of e itself and the old meaning of e. This new meaning assignment is one-to-one, i.e., no two expressions have the same meaning, and that is in fact enough for compositionality in the standard sense (Section 3.7) to be (trivially) satisfied. The compositional extension problem considered here, on the other hand, is quite different. Now we are required to find a total compositional semantics which *agrees* with a given one on certain terms. Such a semantics need not exist. When it does exist, it can be a non-trivial matter to establish that this is so.

Hodges' set-up is based on the notion of a *partial algebra*. In order to make this paper self-contained, I present relevant notions and definitions in Section 3. But before that, in Section 2, I compare his approach to one which is more familiar in formal semantics but uses *many-sorted* algebras instead, and which originates with Montague; see also Section 9.4. Section 4 presents Hodges' Theorem. Section 5 deals with the case when cofinality is dropped but the Husserl property holds. Sections 6 – 8 are devoted to the statement and proof of the main result, and Section 9 ends with some further remarks and problems.

2 Many-Sorted vs. Partial Algebras

Compositionality is the property that the meaning of a complex expression is determined by the meanings of its parts and the 'mode of composition'. To even begin to express this precisely one needs, minimally, a set E of structured expressions and a (perhaps partial) function

$$\mu : E \longrightarrow M$$

from E to some set M of 'meanings'. A natural way to think of expressions as structured is to take E as the carrier (domain) of some kind of algebra, so that expressions are generated by means of the operations ('rules') of the algebra starting from some atoms. In general, nothing prevents an expression from being generated in more than one way, i.e., nothing prevents the occurrence of (structural) *ambiguity*. Therefore one assigns meanings not as above to elements of E, but to *derivations* of these elements, and these can be conveniently identified with *terms* in a corresponding *term algebra* $T_{\mathbf{E}}$. Expressions can be thought of as surface strings, which are the *values* of the terms in the term algebra; an ambiguous string is the value of more than one term.

$$\begin{tikzcd} E & & M \\ & T_{\mathbf{E}} \arrow[ul, "val"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

In practice, almost all grammars proposed for natural languages use some system of *categories* to classify expressions, and to constrain the arguments and values of the rules, i.e., the operations of the algebras. One idea is then to take these algebras as *many-sorted*, where sorts correspond to syntactic categories. This goes back to Montague [14] (though he didn't use the notion of a many-sorted algebra) and was developed by Janssen [11]; the account sketched below is from Hendriks [5]. The syntax of a language is given by an algebra

$$\mathbf{A} = \langle (A_s)_{s\in S}, (F_\gamma)_{\gamma\in\Gamma}\rangle,$$

where A_s contains the expressions (strings) of sort s (so $E = \bigcup_{s\in S} A_s$), and each F_γ is a total operation among expressions with arguments and values of fixed sorts (given by $\mathbf{A}'$s *signature*). $\mathbf{A}$ is assumed to be *generated*: Each A_s has a subset X_s of *atomic* expressions of that sort (X_s may be empty), and each expression is either atomic or the value of some operator applied to some arguments.

Meanings are thus assigned not to expressions but to terms in a term algebra $T(\mathbf{A})$ corresponding to $\mathbf{A}$ (with the same sorts and signature). The meanings themselves are given by another algebra

$$\mathbf{B} = \langle (B_t)_{t\in T}, (G_\delta)_{\delta\in\Delta}\rangle.$$

The semantic signature can be completely different from the syntactic one, so one considers mappings $\sigma : S \to T$ and $\rho : \Gamma \to \Delta$ from one to the other, satisfying the condition that if F_γ takes objects of sorts $s_1, \ldots, s_n$ to objects of sort s, then $G_{\rho(\gamma)}$ takes objects of sorts $\sigma(s_1), \ldots, \sigma(s_n)$ to objects of sort $\sigma(s)$. Then, a meaning assignment h to terms in $T(\mathbf{A})$ is a (σ, ρ)-*homomorphism* if

(i) $p \in A_s$ implies that $h(p) \in B_{\sigma(s)}$,

(ii) $h(F_\gamma(p_1, \ldots, p_n)) = G_{\rho(\gamma)}(h(p_1), \ldots, h(p_n))$.

Compositionality now amounts to the existence of such a homomorphism.

The framework also allows for the fact that meanings are often provided via an intermediate logical language L; then $\mathbf{B}$ can be the syntactic algebra of L (L is unambiguous so we don't need the term algebra of $\mathbf{B}$), and a homomorphic mapping l from $\mathbf{B}$ to a 'model-theoretic' algebra $\mathbf{M}$ (same sorts and signature as $\mathbf{B}$) gives the semantics of L:

$$T(\mathbf{A}) \ \xrightarrow{h}\ \mathbf{B} \ \xrightarrow{l}\ \mathbf{M}$$

A slight twist is that $\mathbf{B}$ in practice need not have primitive operators corresponding to those in $\mathbf{A}$; instead they are *definable* in $\mathbf{B}$, so one uses what Hendriks calls the polynomial closure $\Pi(\mathbf{B})$ rather than $\mathbf{B}$.

This by now classical type of framework allows modeling of a lot of syntactic and semantic detail (see e.g. Hendriks [4]). On the other hand, for certain purposes – such as a study of compositionality – some of these details may be irrelevant, and even in the way. I have sketched it here mainly to contrast it with the simpler set-up in Hodges [9], which uses *partial* algebras instead. Hodges' account differs from the classical one in the following ways:

- Rather than using sorts or categories to enforce that operations are only to be applied to certain arguments, one lets the operations be *undefined* for unwanted arguments. The result is a (generated) partial algebra without sorts. However, a notion of category can nevertheless be reconstructed via substitutibility.
- Structure among 'meanings' is disregarded. They just form a set, not an algebra whose sorts and signature need to be related to the syntax. Indeed, nothing at all is assumed about these 'meanings'.
- Similarly, although the syntactic partial algebra is called a *grammar* and it is often helpful to think of its objects as strings, nothing is assumed about these objects, nor about the operations on them.[3]
- Term algebras are used much as before, but meaning assignments to terms are allowed to be partial. This relaxes the homomorphism requirement, but leaves the essence of compositionality intact.

Thus, partiality enters at two places: algebras are partial, but also functions associating meanings with terms. Partial algebras have been studied in Universal Algebra (cf. Grätzer [3], Ch. 2), but the generalization of the notion of homomorphism to partial functions gives rise to some apparently new questions, in particular the extension problem introduced by Hodges. We now present the algebraic framework needed to set the stage for that problem.

3 The Framework

Most of the definitions in this section are just as in Hodges [9], with some exceptions and extensions to be noted.

3.1 Grammars

Definition 1 A *grammar* is a partial algebra

$$\mathbf{E} = \langle E, A, \underline{\alpha} \rangle_{\alpha \in \Sigma},$$

[3]That is, nothing needs to be assumed about them in the present work. Actual grammars clearly satisfy a number of constraints, which may be formulated as axioms for the class of algebras one is interested in. For example, Kracht [13], and Keenan and Stabler [12], which both construe grammars as partial algebras (though in different ways), presuppose such extra structure. But it is not needed for the results here.

where E is a set of *expressions*, $A \subseteq E$ a set of *atoms*, and each symbol α in the signature Σ denotes a partial function $\underline{\alpha}$ – called a *syntactic rule* – from E^n to E for some $n \geq 0$.[4] E is assumed to be *generated* from A by the rules. If $[X]_{\mathbf{B}}$, or just $[X]$, is the set generated in an algebra $\mathbf{B}$ from a subset X of its domain, we may thus write $\mathbf{E} = \langle [A], \underline{\alpha} \rangle_{\alpha \in \Sigma}$.[5]

Expressions may be thought of as strings, but no such assumption about them is used below. In fact, they will play a subordinate role, since we are mostly interested in the *terms* that denote them. Let $Var = \{x, y, \dots\}$ be a countable set of variables, disjoint from E.

Definition 2 The set $T(E)$ of *terms* is defined as follows:

- Elements of $Var \cup A$ are *terms*.
- If $t_1, \dots, t_n$ are *terms* and $\alpha \in \Sigma$ is n-ary, then '$\alpha(t_1, \dots, t_n)$' is a *term*.

Terms with variables are used as a convenient means to describe substitution; Sections 3.2 and 3.3 below. $T(E)$ is the domain of a total *term algebra* $T(\mathbf{E})$ *over* $Var \cup A$, whose operations $\alpha^{T(\mathbf{E})}$ from n-tuples of terms to terms are defined in the obvious way. Since $T(E)$ is generated from $Var \cup A$ in $T(\mathbf{E})$, we have

$$T(\mathbf{E}) = \langle [Var \cup A], \alpha^{T(\mathbf{E})} \rangle_{\alpha \in \Sigma}.$$

The interesting (variable-free) terms are those respecting the partiality of $\mathbf{E}$. These are called *grammatical terms*. For example, '$\alpha(b, \beta(c))$' is grammatical only if $\underline{\beta}(c)$ and $\underline{\alpha}(b, \underline{\beta}(c))$ are both defined. The set $GT(E)$ of grammatical terms and the function val, associating with each such term its value in E, are defined simultaneously as follows:

Definition 3

- If $a \in A$ then a is a *grammatical term* and $val(a) = a$.
- If $p_1, \dots, p_n$ are *grammatical terms* with $val(p_i) = e_i$ for $1 \leq i \leq n$, α is n-ary, and $e = \underline{\alpha}(e_1, \dots, e_n)$ is defined, then the term

  $$\alpha(p_1, \dots, p_n)$$

  is also a *grammatical term*, and $val(\alpha(p_1, \dots, p_n)) = e$.

Think of $p \in GT(E)$ as a derivation (analysis tree) of the expression $val(p)$. An expression may have different derivations, in which case there is a structural ambiguity.

[4] Hodges' notation does not distinguish $\underline{\alpha}$ from α. A more standard algebraic notation for $\underline{\alpha}$ would be $\alpha^{\mathbf{E}}$.

[5] In algebra one often takes atoms to be 0-ary operations. This is not necessarily a good idea in a linguistic context, where the atoms are lexical items from which the language is generated. 0-ary operations are by definition *invariant under automorphisms*. Keenan and Stabler [12] show that such invariance (and more restricted versions of it) is a very significant phenomenon in grammars of natural languages, but it is not a property of most lexical items.

$GT(E)$ is the domain of a *partial term algebra* $GT(\mathbf{E})$ *over* A, where, for $p_1, \dots, p_n \in GT(E)$, $\alpha^{GT(\mathbf{E})}(p_1, \dots, p_n)$ is equal to '$\alpha(p_1, \dots, p_n)$' if this term is in $GT(E)$; otherwise undefined.[6] Thus,

$$GT(\mathbf{E}) = \langle [A], \alpha^{GT(\mathbf{E})} \rangle_{\alpha \in \Sigma}.$$

The notion of a *subterm* of a term in $T(E)$ is defined in the usual way. Note that if a term is grammatical, so are all of its subterms.

From now on, $\mathbf{E}$ will be a fixed but arbitrary grammar. We then write simply T (GT) for the corresponding set of (grammatical) terms.

3.2 Substitution

If s is a term (in T), $x_1, \dots, x_n$ are distinct variables, and $p_1, \dots, p_n$ are (not necessarily distinct) terms, then

(1) $$s(p_1, \dots, p_n | x_1, \dots, x_n)$$

denotes the term which results from replacing all occurrences of x_i in s by p_i. This notation for substitution is more fine-grained than the usual one (like '$s[p/q]$'), since it can indicate precisely which *occurrences* of terms that are to be replaced.

We follow the convention that in (1), no other variables than those displayed occur in s. In particular, if $p_1, \dots, p_n$ are grammatical terms, then $s(p_1, \dots, p_n | x_1, \dots, x_n)$ is a variable-free term in T (which may or may not be grammatical).

3.3 Occurrences

One sometimes needs to look at *occurrences* of terms rather than just terms. This is not done in Hodges [9], but the notation (1) can be applied in this case too, provided the following convention is observed.

> *Variable Convention*: When the notation (1) is used, it is presupposed that each variable x_i occurs at most once in s.

This is no loss of generality, since if there are several occurrences of the same variable, they can be replaced by new ones, and the term repeated accordingly in the list $p_1, \dots, p_n$. Then, each variable in (1) that occurs in s marks a unique occurrence of a subterm of $s(p_1, \dots, p_n | x_1, \dots, x_n)$. So we can use (1) also for the case when 'p_1', ..., 'p_n' stand for occurrences of subterms. This should lead to no confusion, as long as it is clearly indicated that we are talking about occurrences rather than terms. The subterm relation extends in an obvious way to occurrences.

[6] There might be some reluctance to call $GT(\mathbf{E})$ a (partial) term algebra, since its algebraic structure depends crucially on the algebra $\mathbf{E}$, whereas the algebraic structure of a standard (total) term algebra like $T(\mathbf{E})$ depends only on the *signature* (Σ) of $\mathbf{E}$. Still, this terminology (not used by Hodges) seems natural enough.

I will rely on an informal understanding of the notion of an occurrence, but nevertheless make the following remarks.[7] Note first that an occurrence is always an occurrence *of* a term, say p, *in* a term, say t, i.e., p is a subterm of t, so that $t = s(p|x)$ for some s. If this occurrence is replaced by another term q (NB q is a term, not an occurrence – it makes no sense to talk about replacing an occurrence with another occurrence), the resulting term $s(q|x)$ will contain a corresponding occurrence of q. So the terms which occur in (have occurrences in) a term t are exactly the subterms of t, including t itself.

In the ternary relation 'X is an occurrence of Y in Z', X is an occurrence, Y is a term, and Z is either a term or an occurrence of a term; the latter is clear from the second of the two facts/stipulations below. The first of these says that it is uniquely determined which term an occurrence is an occurrence of. The second is a form of transitivity.

(2) If p' is an occurrence of p_1 in q_1, as well as an occurrence of p_2 in q_2, then $p_1 = p_2$.

(3) If p' is an occurrence of p in q', which in turn is an occurrence of q in s, then p' is an occurrence of p in s.

3.4 Semantics and Synonymies

Definition 4 (a) A *semantics* for $\mathbf{E}$ is a partial function μ from GT to some set M. $p \in GT$ is (μ-)*meaningful* if $p \in dom(\mu)$. μ is *total* if $dom(\mu) = GT$.

(b) A *synonymy* for $\mathbf{E}$ is a partial equivalence relation on GT, i.e., a symmetric and transitive relation $\equiv$ whose domain is a subset of GT. A semantics μ induces the synonymy $\equiv_\mu$ with the same domain:

$$p \equiv_\mu q \iff p, q \in dom(\mu) \text{ and } \mu(p) = \mu(q).$$

(c) Two semantics μ and ν for $\mathbf{E}$ are *equivalent* if $\equiv_\mu \, = \, \equiv_\nu$.

Here we are mainly interested in semantics up to equivalence. This means that, whenever convenient, we can use synonymies instead. Each synonymy $\equiv$ has a corresponding *equivalence class semantics* $\mu^\equiv$:

$$\mu^\equiv(p) = \{q : p \equiv q\}$$

for $p \in dom(\equiv)$. The synonymy induced by $\mu^\equiv$ is again $\equiv$ (Hodges [9], Lemma 1). So up to equivalence we can talk of *the* semantics corresponding to a given synonymy.

We made no assumptions whatever about what the elements of M – the 'meanings' – are. If we restrict attention to synonymies, M drops out of the picture altogether.

[7] The term/occurrence distinction is not the same as the type/token distinction. Occurrences are not types, since, for example, there are two occurrences of (the type) 'a' in (the type) '$\alpha(a, \beta(a))$'. And they are not tokens either, since tokens exist in space and time, whereas writing '$\alpha(a, \beta(a))$' twice does not give us two new occurrences of 'a' in '$\alpha(a, \beta(a))$', although it does give us two new tokens of that symbol.

3.5 Extensions and Refinements

Our aim is to extend partial semantics to total ones, so we had better explain what 'extend' means. This is clear for partial functions: ν extends μ iff they agree on $dom(\mu)$ and $dom(\mu) \subseteq dom(\nu)$; in other words, iff $\mu \subseteq \nu$.

For synonymies there are (at least) two notions, but no really established terminology. We use the following, when $\equiv$ and $\equiv'$ are synonymies for **E**, and $dom(\mu) = X$:

- $\equiv'$ *includes* $\equiv$ iff $\equiv \subseteq \equiv'$. [When $\equiv$ is $\equiv_\mu$ and $\equiv'$ is $\equiv_\nu$, Hodges writes this $\mu \leq \nu$.]
- $\equiv'$ *extends* $\equiv$ iff $\equiv \; = \; \equiv' \cap (X \times X)$.

If $\equiv'$ extends $\equiv$, it changes nothing on the terms in X. If it merely includes $\equiv$, it may introduce new synonymies among 'old' terms. This terminology is apt, since extension for synonymies corresponds to extension for semantics (Hodges [9], Lemma 2):

(4) $\equiv_\nu$ extends $\equiv_\mu$ iff ν is equivalent to some extension of μ.

A related and also useful notion (implicit in [9]) is that of refinement:

- $\equiv'$ *refines* $\equiv$ iff (i) $dom(\equiv) \subseteq dom(\equiv')$; and (ii) $\equiv' \cap (X \times X) \subseteq \equiv$.

Thus, if $\equiv'$ extends $\equiv$, it refines (and includes) $\equiv$. Merely refining $\equiv$ allows making finer meaning distinctions among 'old' terms, while keeping all the distinctions that $\equiv$ makes.

3.6 Semantic Categories

For $X \subseteq GT$, and $p, q \in GT$, define

$$p \sim_X q \text{ iff for all terms } s \in T,\ s(p|x) \in X \Leftrightarrow s(q|x) \in X.$$

This gives us various notions of 'categories', as equivalence classes of $\sim_X$. When $X = dom(\mu)$, we call these μ-categories, or *semantic* categories (substitution preserves meaningfulness). When $X = GT$, it makes sense to talk about *syntactic* categories (preservation of grammaticality). Syntactic categories induce a corresponding partition of the set E of expressions via the function *val* since, as one easily shows,

(5) $$val(p) = val(q) \text{ implies } p \sim_{GT} q.$$

In the many-sorted algebra framework, on the other hand, no assumptions of non-emptiness or disjointness are made about the carriers of the various sorts. In this sense, the many-sorted approach allows more freedom than the partial one – but see Section 9.4.

Definition 5 μ has the *Husserl property* if $p \equiv_\mu q$ implies $p \sim_{dom(\mu)} q$.

The Husserl property, that synonymous terms have the same semantic category, is reasonable in many contexts. It is, however, quite a strong assumption on the semantics, and it is not always obvious in applications that it should be expected to hold (see Hodges [9], Section 4). So it is natural to inquire what happens without it.[8]

3.7 Compositionality

Hodges presents several versions of compositionality; here is the basic one:

Definition 6 Let Comp($\equiv_\mu$) be the following condition:

(6) For any n, if $p_i \equiv_\mu q_i$ for $1 \leq i \leq n$, and if $s(p_1, \dots, p_n|x_1, \dots, x_n)$, $s(q_1, \dots, q_n|x_1, \dots, x_n)$ are both in $dom(\equiv_\mu)$, then

$$s(p_1, \dots, p_n|x_1, \dots, x_n) \equiv_\mu s(q_1, \dots, q_n|x_1, \dots, x_n).$$

Also, let 1-Comp($\equiv_\mu$) be the same condition but with $n = 1$. Similarly for any synonymy $\equiv$ for $\mathbf{E}$.

If μ is husserlian, 1-Comp($\equiv_\mu$) implies Comp($\equiv_\mu$), since we can then replace p_i by q_i one by one, and be sure that the 'intermediate' terms arising in this process are all meaningful. But without the Husserl property, 1-Comp($\equiv_\mu$) is strictly weaker.

The formulation of compositionality which corresponds most directly to the intuitive idea – i.e., that the meaning of complex term is determined by the meanings of its parts and the 'mode of composition' – is the following one, which we may call Rule(μ):

(7) For each $\alpha \in \Sigma$, there is an operation r_α such that whenever $\alpha(p_1, \dots, p_k)$ is in the domain of μ,

$$\mu(\alpha(p_1, \dots, p_k)) = r_\alpha(\mu(p_1), \dots, \mu(p_k)).$$

Note, however, that this only makes sense if $dom(\mu)$ is closed under subterms. But under this condition, as Hodges shows, Comp($\equiv_\mu$) and Rule(μ) are equivalent.

This ends my exposition of Hodges' framework. Observe that Comp($\equiv_\mu$) looks similar to the more familiar requirement that $\equiv_\mu$ is a *congruence relation*, modulo the restriction to $dom(\mu)$. Indeed, consider the following variant, that I will call Congr($\equiv_\mu$):

[8] An example: Presumably the English words *run* and *runs* (taken as forms of the verb in present tense) are synonymous, but are they intersubstitutable? This depends on what we mean by 'substitution' and on how the rules are made. If we replace *runs* by *run* in *John runs* we get something ungrammatical. So, first, a substitution in the sense relevant here only takes place at the input of a grammar rule. Thus, second, if *run* and *runs* are inputs to the same rule, the Husserl property is violated. But, third, it is not necessary for the rule to behave like this. Instead, it could apply to the infinitival form (*run*), and add the *-s* if the other argument (in this case *John*) is in the 3rd person singular. Now the Husserl property holds.

(8) If $p_i \equiv_\mu q_i$ for $1 \leq i \leq k$, $\alpha \in \Sigma$ is k-ary, and $\alpha(p_1, \dots, p_k)$, $\alpha(q_1, \dots, q_k)$ are both in $dom(\mu)$, then $\alpha(p_1, \dots, p_k) \equiv_\mu \alpha(q_1, \dots, q_k)$.

Fact 7 *If $dom(\mu)$ is closed under subterms, then Comp($\equiv_\mu$) is equivalent to Congr($\equiv_\mu$).*

Proof. It is immediate that Comp($\equiv_\mu$) implies Congr($\equiv_\mu$): just take $s = \alpha(x_1, \dots, x_k)$. In the other direction, prove (6) by induction on s. The basis step is when $s \in Var \cup A$. If $s \in A$ or if s is a variable $\neq x_1, \dots, x_n$ then $s(p_1, \dots, p_n|\dots) = s(q_1, \dots, q_n|\dots)$. If $s = x_i$ then $s(p_1, \dots, p_n|\dots) = p_i$ and $s(q_1, \dots, q_n|\dots) = q_i$. In both cases (6) holds.

For the inductive step, when $s = \alpha(t_1, \dots, t_k)$, it is assumed that

$$s(p_1, \dots, p_n|\dots) = \alpha(t_1(p_1, \dots, p_n|\dots), \dots, t_k(p_1, \dots, p_n|\dots))$$

is in $dom(\mu)$, and likewise for $s(q_1, \dots, q_n|\dots)$. To be able to conclude from the induction hypothesis that

$$t_i(p_1, \dots, p_n|\dots) \equiv_\mu t_i(q_1, \dots, q_n|\dots),$$

we need to know that both terms are in $dom(\mu)$, and this follows from the assumptions and the fact that $dom(\mu)$ is closed under subterms. Then (6) follows from Congr($\equiv_\mu$). □

On the other hand, if $dom(\mu)$ is not closed under subterms, we could easily have $a \equiv_\mu b$, $\alpha(a), \alpha(b) \notin dom(\mu)$, $\beta(\alpha(a)), \beta(\alpha(b)) \in dom(\mu)$, but $\beta(\alpha(a)) \not\equiv_\mu \beta(\alpha(b))$, which is compatible with Congr($\equiv_\mu$) but not with Comp($\equiv_\mu$). In some applications one doesn't want to assume closure under subterms, so Hodges' condition Comp($\equiv_\mu$) is the correct most general version of compositionality. On the other hand, Congr($\equiv$) makes sense for any partial equivalence relation $\equiv$ on any partial algebra, not just for term algebras.

3.8 The Compositional Extension Problem

Let μ be a compositional semantics (for our fixed grammar) with $dom(\mu) = X \subseteq GT$. The general problem that interests us is:

(9) When does μ have a total compositional extension?

As noted earlier, it is equivalent to ask this question for $\equiv_\mu$ instead, but recall (Section 3.5) that 'extend' then means something much stronger than 'include': *Any* partial synonymy, compositional or not, trivially has a total compositional synonymy that includes it, namely, $GT \times GT$!

The next example shows that the compositional extension problem is not quite trivial.

Example 8 Suppose all of the following are true:

$$a \equiv_\mu b, \quad \alpha(b) \equiv_\mu c$$

$$\beta(\alpha(a)), \beta(c) \in X$$

$$\alpha(a), \beta(\alpha(b)) \in GT - X$$

$$\beta(\alpha(a)) \not\equiv_\mu \beta(c)$$

It is easy to find a compositional semantics μ for which this holds: the non-synonymy of $\beta(\alpha(a))$ and $\beta(c)$ does not contradict compositionality since the relevant subterms are not μ-meaningful. But there is no compositional extension of μ to these terms, *a fortiori* no total one. □

The semantics in this example is certainly not husserlian (for example, $\alpha(b) \in X$ but $\alpha(a) \in GT - X$, even though $a \equiv_\mu b$), and neither is its domain closed under subterms.

4 Hodges' Theorem

I said in Section 1 that Hodges proves his main result under two extra conditions, one of which is the Husserl property. But not all of his observations use these conditions. The formulation below (which draws on [10], though the claims can be dug out of [9] as well) exhibits exactly what each claim requires.

The key notion corresponds to Frege's 'Contribution Principle' (F) mentioned in the Introduction:

Definition 9 Suppose $dom(\mu) = X \subseteq GT$. A relation $\equiv$ on GT is a (total) *fregean cover* of $\equiv_\mu$ iff it is a total synonymy on GT and the following holds:

F(a): $p \equiv q$ and $t(p|x) \in X$ implies $t(q|x) \in X$.

F(b): $p \equiv q$ and $t(p|x), t(q|x) \in X$ implies that $t(p|x) \equiv_\mu t(q|x)$.

F(c): If $p \not\equiv q$ there is a term t such that either exactly one of $t(p|x), t(q|x)$ is in X, or both are and $t(p|x) \not\equiv_\mu t(q|x)$.

$\equiv$ is a *fregean extension* of $\equiv_\mu$ if it in addition extends $\equiv_\mu$.

F(a) is a version of the Husserl property for two synonymies, and F(b) is a corresponding version of 1-Comp. F(c) is a converse to 1-Comp, related to what computer scientists call 'full abstraction'. Now (F) says that the meaning of any term is determined by how it contributes to the meanings of terms in X of which it is a part. When this is expressed in terms of synonymies instead, the vague term "contribute" disappears: we can say that the fregean cover must not make other distinctions than those warranted by the given synonymy – this is F(c) – and that it must make all those that are warranted – F(a) and F(b). Of course it remains to be shown that an extension of $\equiv_\mu$ with these properties exists.

With $\equiv_\mu$ as in the above definition, define the relation $\equiv_\mu^{\mathrm{F}}$ as follows:

(10) $p \equiv_\mu^{\mathrm{F}} q$ iff $p \sim_X q$, and for all s, if $s(p|x) \in X$ then $s(p|x) \equiv_\mu s(q|x)$,

for $p, q \in GT$. We collect most of Hodges' results in the following theorem.

Theorem 10 (Hodges' Theorem) *Suppose $dom(\mu) = X \subseteq GT$.*

(i) *There exists a unique fregean cover of $\equiv_\mu$, namely, $\equiv_\mu^{\mathrm{F}}$. Moreover, $\equiv_\mu^{\mathrm{F}}$ refines $\equiv_\mu$.*

(ii) *If X is cofinal in GT (i.e., every term in GT is a subterm of some term in X), then $\equiv_\mu^{\mathrm{F}}$ is husserlian and compositional.*

(iii) (Extension Theorem) $\equiv_\mu^{\mathrm{F}}$ *extends* $\equiv_\mu$ *iff* $\equiv_\mu$ *is husserlian and compositional.*

(iv) *These facts, and the definition of a fregean cover, generalize to the case when GT is replaced by any set Y such that $X \subseteq Y \subseteq GT$ and Y is closed under subterms.*

Proof. (Outline) (i) Let us verify uniqueness, since this illustrates nicely how the notion of a fregean cover works. So suppose $\equiv_1$ and $\equiv_2$ are fregean covers of $\equiv_\mu$, and that $p \equiv_1 q$ but $p \not\equiv_2 q$. By F(c), there is a term t such that either exactly one of $t(p|x), t(q|x)$ is in X, or both are and $t(p|x) \not\equiv_\mu t(q|x)$. But the first case violates F(a) for $\equiv_1$, and the second case violates F(b). This shows that $\equiv_1 \subseteq \equiv_2$. The inverse inclusion follows by symmetry, and thus $\equiv_1 = \equiv_2$.

Next, one checks one by one the conditions for $\equiv_\mu^{\mathrm{F}}$ being a total synonymy such that F(a)–(c) hold. Finally, suppose $p, q \in X$ and $p \equiv_\mu^{\mathrm{F}} q$. Taking s as the term x in (10), it follows that $p \equiv_\mu q$, and hence $\equiv_\mu^{\mathrm{F}}$ refines $\equiv_\mu$.

(ii) The verification is not hard, but uses essentially the cofinality of X and that $\equiv_\mu^{\mathrm{F}}$ satisfies F(a)–(c).

(iii) Suppose $\equiv_\mu$ is husserlian and compositional. By (i), it only remains to show that for $p, q \in X$, $p \equiv_\mu q$ implies $p \equiv_\mu^{\mathrm{F}} q$. But if $p \equiv_\mu q$ we get $p \sim_X q$ by the Husserl property, and the second condition in the right hand side of (10) by compositionality. Thus, $p \equiv_\mu^{\mathrm{F}} q$.

Conversely, suppose $\equiv_\mu^{\mathrm{F}}$ extends $\equiv_\mu$. Thus, if $p \equiv_\mu q$ then $p \equiv_\mu^{\mathrm{F}} q$, so if $s(p|x) \in X$, it follows by F(a) that $s(q|x) \in X$. That is, $\equiv_\mu$ is husserlian. 1-compositionality follows similarly by F(b), and general compositionality is equivalent to 1-compositionality under the Husserl property.

(iv) This follows since it can be seen that the only properties of GT used above are in fact that it is closed under subterms and contains X. □

It follows that any partial, husserlian, and compositional semantics μ whose domain is cofinal in GT has a unique (up to equivalence) total fregean extension, and that this extension also is husserlian and compositional. Then, it is indeed the case that the meaning of any grammatical term is determined (up to equivalence) by the contributions it makes to the meanings of terms in the domain of μ.

5 Dropping Cofinality

Without cofinality, we cannot use Hodges' Theorem to infer that if μ is husserlian and compositional, its fregean cover has the same properties. However, we now show that we still get compositionality in this case.

Corollary 11 *If $\equiv_\mu$ is husserlian and compositional, its total fregean extension $\equiv_\mu^{\mathrm{F}}$ is always compositional, but not necessarily husserlian.*

Proof. We start with Hodges' Theorem, and a construction which is also from [9] (Lemma 9): define

$$Y = \{p \in GT : p \text{ is a subterm of some term in } X\}.$$

Then X is cofinal in Y, and Y is closed under subterms, so by (iv) of Hodges' Theorem there is a fregean extension μ^f of μ to Y, which again is compositional and husserlian.

Now let μ_1 be a *one-point extension* of μ^f to GT, i.e., an extension coinciding with μ^f on Y and making all terms in $GT - Y$ synonymous with each other, but not with anything in Y.

It is easy to see that μ_1 is in fact a fregean extension of μ (Hodges [9], Lemmas 8(a) and 9). By (i) of Hodges' Theorem, $\equiv_{\mu_1} = \equiv_\mu^{\mathrm{F}}$. But without cofinality we cannot conclude that $\equiv_\mu^{\mathrm{F}}$ is husserlian. Indeed, the following situation is consistent with our assumptions:

$$a \equiv_\mu b, \;\; \alpha(a) \in GT - Y, \text{ but } \alpha(b) \notin GT. \tag{11}$$

In this case no total extension of μ can have the Husserl property, even though μ does.

Nevertheless, $\equiv_\mu^{\mathrm{F}}$ is still compositional. For suppose

$$p_i \equiv_\mu^{\mathrm{F}} q_i, \; 1 \leq i \leq n,$$

and that $s(p_1, \dots, p_n|x_1, \dots, x_n)$ and $s(q_1, \dots, q_n|x_1, \dots, x_n)$ are both grammatical. (Since the Husserl property might fail, it is not enough to show 1-compositionality.) If $s(p_1, \dots, p_n|x_1, \dots, x_n)$ and $s(q_1, \dots, q_n|x_1, \dots, x_n)$ are both in $GT - Y$, then they are μ_1-synonymous by definition. So suppose $s(p_1, \dots, p_n|x_1, \dots, x_n) \in Y$. Then $p_1, \dots, p_n \in Y$ since Y is closed under subterms. But then also $q_1, \dots, q_n \in Y$, since if $q_i \in GT - Y$, $p_i \not\equiv_{\mu_1} q_i$ by definition of μ_1. Thus

$$p_i \equiv_{\mu^f} q_i, \; 1 \leq i \leq n.$$

By (n applications of) the Husserl property for μ^f, $s(q_1, \dots, q_n|x_1, \dots, x_n) \in Y$. Therefore

$$s(p_1, \dots, p_n|x_1, \dots, x_n) \equiv_{\mu^f} s(q_1, \dots, q_n|x_1, \dots, x_n)$$

by the compositionality of μ^f, and hence

$$s(p_1, \ldots, p_n|x_1, \ldots, x_n) \equiv_\mu^{\mathrm{F}} s(q_1, \ldots, q_n|x_1, \ldots, x_n).$$

$\square$

6 An Extension Theorem without the Husserl Property

If $\equiv_\mu$ is compositional but lacks the Husserl property, its fregean cover $\equiv_\mu^{\mathrm{F}}$ is no longer of interest when we look for a total compositional extension, since by Hodges' Theorem (iii), $\equiv_\mu^{\mathrm{F}}$ does *not* then extend $\equiv_\mu$. We shall show that under one very weak extra assumption, a total compositional extension must nevertheless exist. This is the property of a semantics, mentioned before, that its domain is closed under subterms. It has been called the *Domain Principle*, and is sometimes taken as part of the notion of compositionality (cf. Section 3.7).

Theorem 12 *If μ is a compositional semantics whose domain is closed under subterms, then $\equiv_\mu$ (and hence μ) has a total compositional extension.*

This follows from Corollary 11 if μ is also husserlian. In fact, in that case it is easy to see, using closure under subterms, that the one-point extension of $\equiv_\mu$ to GT is identical to $\equiv_\mu^{\mathrm{F}}$. But we also saw from the proof of that corollary that not all compositional semantics can be extended to husserlian ones.

Before starting on the proof of Theorem 12, let me make one remark, which underlines the observation that in the absence of the Husserl property it is not enough to consider merely 1-compositionality. First:

(12) There is a 1-compositional semantics (whose domain can be assumed to be either cofinal in GT or to be closed under subterms) with no total compositional extension.

This follows immediately from the existence of a total semantics which is 1-compositional but not compositional (and hence not husserlian). For example, suppose $a_1 \equiv_\mu b_1$, $a_2 \equiv_\mu b_2$, $\alpha(a_1, a_2), \alpha(b_1, b_2) \in dom(\mu)$, $\alpha(a_1, a_2) \not\equiv_\mu \alpha(b_1, b_2)$, but $\alpha(a_1, b_2), \alpha(b_1, a_2) \notin GT$. These assumptions are consistent with μ being total.

By elaborating the example a little we obtain a stronger conclusion.

(13) There is a 1-compositional semantics (whose domain can be assumed to be either cofinal in GT or to be closed under subterms) with no total and 1-compositional extension.

Proof. Let there be six atoms, $A = \{a, b, c_0, c_1, d_0, d_1\}$, and a 3-place rule α. Let

$$\begin{aligned} p_0 &= \alpha(a, c_0, d_0) \\ p_1 &= \alpha(a, c_1, d_0) \\ p_2 &= \alpha(b, c_0, d_0) \\ p_3 &= \alpha(b, c_0, d_1) \end{aligned}$$

and suppose the grammar is such that

$$GT = A \cup \{p_0, p_1, p_2, p_3\}.$$

Now if $X = A \cup \{p_1, p_3\}$, X is closed under subterms. If we instead want X to be a cofinal subset of GT, add a 1-place rule β, and add $\beta(p_0), \beta(p_2)$ to GT and to X.

Now take a semantics μ with domain X such that $\mu(a) = \mu(b)$, $\mu(c_0) = \mu(c_1)$, $\mu(d_0) = \mu(d_1)$, $\mu(p_1) \neq \mu(p_3)$, and (in the second case) $\mu(\beta(p_0)) = \mu(\beta(p_2))$. μ is not husserlian or compositional, but it is trivially 1-compositional, since if we substitute according to one synonymous pair at a time we end up outside X. (In the second case, we have $\beta(p_0) \equiv_\mu \beta(p_2)$, which does not disturb compositionality.) On the other hand,

(14) μ has no total 1-compositional extension.

For suppose μ' were such an extension. Then the synonymies mentioned above together with 1-compositionality imply that $p_0 \equiv_{\mu'} p_1$, $p_2 \equiv_{\mu'} p_3$, and $p_0 \equiv_{\mu'} p_2$. Hence, $p_1 \equiv_{\mu'} p_3$, but this contradicts the assumption that μ' extends μ. $\square$

In preparation for the proof of Theorem 12, we devote the next section to some observations about occurrences of subterms, and to a generalized version of compositionality.

7 A Generalization of Compositionality

It is sometimes necessary to distinguish occurrences of (sub)terms from the terms themselves. Here is a trivial example.[9]

(15) If $p_1, \ldots, p_n$ are distinct occurrences of terms in s, and if

$$s = s_0(p_1, \ldots, p_n | x_1, \ldots, x_n) = t_0(p_1, \ldots, p_n | y_1, \ldots, y_n),$$

then, for any terms $p'_1, \ldots, p'_n$,

$$s_0(p'_1, \ldots, p'_n | x_1, \ldots, x_n) = t_0(p'_1, \ldots, p'_n | y_1, \ldots, y_n).$$

[9] Recall from Section 3.3 that our notation for substitution can be used with occurrences of terms as well as with terms – though preferably not at the same time. On the first displayed line in (15) below it is used in the first way, and on the second line it is used in the second way. Thus, for example, there can be no repetitions in the sequence $p_1, \ldots, p_n$, but there can be in $p'_1, \ldots, p'_n$.

Though trivial, this fails completely if $p_1, \ldots, p_n$ are terms rather than occurrences of terms. For example,

$$\alpha(a, c, a) = \alpha(x, c, a)(a|x) = \alpha(a, c, x)(a|x),$$

but

$$\alpha(b, c, a) = \alpha(x, c, a)(b|x) \neq \alpha(a, c, x)(b|x) = \alpha(a, c, b).$$

The Occurrence Lemma below generalizes (15). We use the following terminology. Suppose s is a term such that

$$s = s_0(p_1, \ldots, p_k|x_1, \ldots, x_k) = t_0(q_1, \ldots, q_n|y_1, \ldots, y_n),$$

where the p_i and q_j are distinct occurrences of terms in s. Thus, no p_i is a subterm of any distinct $p_{i'}$, and similarly for the q_j, though p_i might be a subterm of q_j, or vice versa, for certain i, j. A *subsequence* of $p_1, \ldots, p_k, q_1, \ldots, q_n$ is a sequence containing only occurrences from the sequence $p_1, \ldots, p_k, q_1, \ldots, q_n$ and in the same order.

Lemma 13 (Occurrence Lemma) *Suppose*

(16) $$s = s_0(p_1, \ldots, p_k|x_1, \ldots, x_k) = t_0(q_1, \ldots, q_n|y_1, \ldots, y_n),$$

where the p_i and q_j are distinct occurrences of terms in s. Then there exists a subsequence $r_1, \ldots, r_m$ of $p_1, \ldots, p_k, q_1, \ldots, q_n$ (call it maximal *with respect to* (16)*), such that*

(i) $s = t(r_1, \ldots, r_m|z_1, \ldots, z_m)$ *for some term* t.

(ii) *Each p_i is a subterm of one of $r_1, \ldots, r_m$, and likewise for each q_j.*

(iii) *For $1 \leq i \neq j \leq m$, r_i is not a subterm of r_j.*

Now suppose $p'_1, \ldots, p'_k$ are arbitrary terms. Let r'_i be the result of replacing each occurrence p_j in r_i by p'_j. Then

(a) $s_0(p'_1, \ldots, p'_k|x_1, \ldots, x_k) = t(r'_1, \ldots, r'_m|z_1, \ldots, z_m)$.

Similarly, if $q'_1, \ldots, q'_n$ are terms, and s'_i is the result of replacing each subterm q_j in r_i by q'_j, then

(b) $t_0(q'_1, \ldots, q'_n|y_1, \ldots, y_n) = t(s'_1, \ldots, s'_m|z_1, \ldots, z_m)$.

We remark that r_i in the lemma may have no occurrences at all of the p_j, in which case $r'_i = r_i$. Or r_i could have a subterm from $p_1, \ldots, p_k$ but no proper one. Then $r_i = p_j$ for some j, and $r'_i = p'_j$. In this case it is also possible that $r_i = p_j = q_l$ for some l. If instead r_i has at least one proper subterm from $p_1, \ldots, p_k$, then for some l, $r_i = q_l$ and is of the form $u(p_{l_1}, \ldots, p_{l_{k_l}}|y_{l_1}, \ldots, y_{l_{k_l}})$, so $r'_i = u(p'_{l_1}, \ldots, p'_{l_{k_l}}|y_{l_1}, \ldots, y_{l_{k_l}})$. Similar remarks apply to s'_i.

Note also that (15) is the special case of the Occurrence Lemma when $k = n$ and $p_i = q_i$, for then $m = k$ and $r_i = p_i$ (so $r'_i = p'_i$), and we can let $t = t_0$, so (15) is just (a).

The Occurrence Lemma is in fact an immediate observation, once one sees what is going on. And to see this, a sufficiently typical example will be enough:

Example 14 Let

$$s \;=\; \gamma(p_1, \underbrace{\alpha(q_1, q_2)}_{p_2}, \underbrace{\beta(p_3, p_4, p_5)}_{q_3}, \underbrace{q_4}_{p_6}) \;=\; \gamma(r_1, \ldots, r_4);$$

so, for the obvious s_0 and t_0,

$$s \;=\; s_0(p_1, \ldots, p_6 | x_1, \ldots, x_6) \;=\; t_0(q_1, \ldots, q_4 | y_1, \ldots, y_4).$$

Note that $r_4 = q_4 = p_6$. $r_1, \ldots, r_4$ is a maximal subsequence of the sequence $p_1, \ldots, p_6, q_1, \ldots, q_4$. (So $p_1, \ldots, p_6, q_1, \ldots, q_4$ contains a repetition, and $r_1, \ldots, r_4$ could be either one of p_1, p_2, q_3, q_4 and p_1, p_2, p_6, q_3; it does not matter which.) And we see that, as described in the Occurrence Lemma, for any terms $p'_1, \ldots, p'_6$ and $q'_1, \ldots, q'_4$,

$$s_0(p'_1, \ldots, p'_6 | x_1, \ldots, x_6) \;=\; \gamma(r'_1, \ldots, r'_4) \;=\; \gamma(p'_1, p'_2, \beta(p'_3, p'_4, p'_5), p'_6)$$

and

$$t_0(q'_1, \ldots, q'_4 | y_1, \ldots, y_4) \;=\; \gamma(s'_1, \ldots, s'_4) \;=\; \gamma(p_1, \alpha(q'_1, q'_2), q'_3, q'_4).$$

We now apply the Occurrence Lemma to compositionality.

Lemma 15 (Generalized Compositionality Lemma) *Suppose μ is a compositional semantics (for our fixed grammar $\mathbf{E}$) such that $X = dom(\mu)$ is closed under subterms, and suppose*

$$s_0(p_1, \ldots, p_k | x_1, \ldots, x_k) = t_0(q_1, \ldots, q_n | y_1, \ldots, y_n)$$

is a grammatical term. If $p_i \equiv_\mu p'_i$ for $1 \le i \le k$ and $q_j \equiv_\mu q'_j$ for $1 \le j \le n$, and if $s_0(p'_1, \ldots, p'_k | x_1, \ldots, x_k)$ and $t_0(q'_1, \ldots, q'_n | y_1, \ldots, y_n)$ are both in X, then

$$s_0(p'_1, \ldots, p'_k | x_1, \ldots, x_k) \equiv_\mu t_0(q'_1, \ldots, q'_n | y_1, \ldots, y_n).$$

Proof. In this result it is not presupposed that $p_1, \ldots, p_k$ and $q_1, \ldots, q_n$ are distinct occurrences of terms. The first thing to observe, however, is that we might as well assume that they are. In fact, many claims about grammars and semantics have a standard version and an 'ocurrence version', which turn out to be equivalent – another example is the principle of compositionality itself, Comp($\equiv_\mu$). The proof of the required equivalence is completely straightforward (using the facts about occurrences mentioned in Section 3.3 and the Occurrence Lemma) but somewhat tedious, and I shall spare the reader.

Thus, suppose the assumptions in the lemma hold, with $p_1, \dots, p_k$ and $q_1, \dots, q_n$ distinct occurrences of terms, and let r'_i, s'_i, $1 \leq i \leq m$, be as in the Occurrence Lemma. We claim that

$$r'_i \equiv_\mu s'_i$$

for $1 \leq i \leq m$. By the Occurrence Lemma, compositionality, and the assumption that $s_0(p'_1, \dots, p'_k|x_1, \dots, x_k)$ and $t_0(q'_1, \dots, q'_n|y_1, \dots, y_n)$ are both in X, the desired conclusion follows.

To prove the claim, consider first the case when $r_i = p_j$ for some j. Then $r'_i = p'_j$, and moreover we have

$$r_i = u(q_{l_1}, \dots, q_{l_{k_l}}|w_{l_1}, \dots, w_{l_{k_l}})$$

for some term u. (This covers the case when $r_i = q_v$ (with $k_l = 1$ and $u = y_1$), as well as the case when r_i has no occurrences at all of $q_1, \dots, q_n$ (with $u = p_j$)). Now we get

$$r'_i = p'_j \equiv_\mu p_j = u(q_{l_1}, \dots, q_{l_{k_l}}|\dots) \equiv_\mu u(q'_{l_1}, \dots, q'_{l_{k_l}}|\dots) = s'_i.$$

The first μ-equivalence here is an assumption. The second follows from compositionality, once we know that the two terms are in X. But $p_j \in X$ by assumption, and $s'_i \in X$ since, by (b) in the Occurrence Lemma, it is a subterm of $t_0(q'_1, \dots, q'_n|y_1, \dots, y_n) \in X$, and X is closed under subterms.

If $r_i \neq p_1, \dots, p_k$ then it must be equal to q_j for some j. This case is symmetric to the previous one. The proof is complete. □

The point of the Generalized Compositionality Lemma is that the grammatical term $s_0(p_1, \dots, p_k|x_1, \dots, x_k)$ $(= t_0(q_1, \dots, q_n|y_1, \dots, y_n))$ need not be in the domain of μ. If $s_0(p_1, \dots, p_k|x_1, \dots, x_k) \in dom(\mu)$, the conclusion of the lemma is immediate by two applications of compositionality; indeed it would then suffice to assume $s_0(p_1, \dots, p_k|x_1, \dots, x_k) \equiv_\mu t_0(q_1, \dots, q_n|y_1, \dots, y_n)$, and closure of $dom(\mu)$ under subterms would not be required. But in the next section it will be essential to consider the more general situation described in the lemma.

8 Proof of Theorem 12

Suppose $\equiv_\mu$ is compositional and $X = dom(\mu)$ is closed under subterms. To extend $\equiv_\mu$ to all of GT, new pairs may need to be added. Suppose, for example, that $\alpha(a) \equiv_\mu \beta(c_1, c_2)$, $a \equiv_\mu b$, $c_i \equiv_\mu d_i$, $i = 1, 2$, but $\alpha(b), \beta(d_1, d_2) \in GT - X$. Then clearly it needs to be the case that $\alpha(b)$ is equivalent to $\beta(d_1, d_2)$ (and of course $\alpha(a)$ to $\alpha(b)$ etc.) in the extended semantics. So the idea is simply to add such pairs to the synonymy $\equiv_\mu$.

The new synonymies generate further requirements, so the above step has to be repeated ω times. This will insure a weaker version of the Husserl property (we know the full version cannot be obtained in general), which is still such that a final one-point extension gives the desired total semantics. Of course one must check that compositionality, closure under subterms, etc., are preserved at each step. The next subsection is devoted to the main step indicated above.

8.1 Extension to Corresponding Terms

Forget for the moment about μ, and let (in this subsection) $\equiv$ be any synonymy for $\mathbf{E}$, with $X = dom(\equiv)$.

Let $s, s' \in GT$. We say that s *corresponds* to s' if s' results from substituting $\equiv$-equivalent subterms in s. Then we let $s \equiv^+ t$ if s and t correspond to $\equiv$-equivalent terms in X. In more detail:

Definition 16 (a) *s corresponds* to s' (relative to $\equiv$) if there is a term s_0 and distinct occurrences $p_1, \dots, p_k$ in s and $p'_1, \dots, p'_k$ in s' such that $p_i \equiv p'_i$ for $1 \le i \le k$, and

$$s = s_0(p_1, \dots, p_k | x_1, \dots, x_k) \text{ and } s' = s_0(p'_1, \dots, p'_k | x_1, \dots, x_k).$$

We let X^+ be the set of terms corresponding to terms in X. Thus (since any term in X corresponds to itself), $X \subseteq X^+ \subseteq GT$.

(b) For $s, t \in GT$, let

$$s \equiv^+ t$$

iff there is a term $s' \in X$ corresponding to s and a term $t' \in X$ corresponding to t such that $s' \equiv t'$.

Lemma 17 *$\equiv^+$ is symmetric, and reflexive on its field X^+. Also, if s corresponds to $s' \in X$, then $s \equiv^+ s'$.*

Proof. Immediate. □

Lemma 18 *If $\equiv$ is compositional, then $\equiv^+$ extends $\equiv$, i.e., for all $s, t \in X$, $s \equiv t \Leftrightarrow s \equiv^+ t$.*

Proof. Suppose $s, t \in X$. If $s \equiv t$ clearly $s \equiv^+ t$. If $s \equiv^+ t$, let s', t' correspond to s, t, respectively, as in Definition 16 (b). Since $s, t \in X$, $\equiv$-compositionality applies, so $s \equiv s' \equiv t' \equiv t$. □

Lemma 19 *If X is closed under subterms, so is X^+.*

Proof. Suppose $s \in X^+$ and let q be a subterm of s. It is clearly no loss of generality here to assume that q is an *occurrence* of a term in s. Thus we have

$$s = s_0(p_1, \ldots, p_k|x_1, \ldots, x_k) = t_0(q|y),$$

and also $p'_1, \ldots, p'_k$ such that

$$s_0(p'_1, \ldots, p'_k|x_1, \ldots, x_k) \in X$$

and $p_i \equiv p'_i$, $1 \leq i \leq k$ (where $p_1, \ldots, p_k$ and $p'_1, \ldots, p'_k$ are occurrences in the respective terms). Let $r_1, \ldots, r_m$ be a maximal sequence relative to $p_1, \ldots, p_k, q$ and s as in the Occurrence Lemma (Lemma 13), so that

$$s = t(r_1, \ldots, r_m|z_1, \ldots, z_m).$$

Thus,

$$s_0(p'_1, \ldots, p'_k|x_1, \ldots, x_k) = t(r'_1, \ldots, r'_m|z_1, \ldots, z_m) \in X, \tag{17}$$

where r'_i is the result of replacing each occurrence p_j in r_i by p'_j.

Case 1. q is a subterm of some p_j. Then $q \in X$, since X is closed under subterms, so $q \in X^+$.

Case 2. q is not a subterm of any $p_1, \ldots, p_k$. Then $q = r_i$ for some i.

Subcase 2.1. q does not overlap with any of $p_1, \ldots, p_k$. Then $r'_i = q$, and r'_i is a subterm of $t(r'_1, \ldots, r'_m|z_1, \ldots, z_m) \in X$, so again $q \in X$.

Subcase 2.2. Not Subcase 2.1. Then

$$q = r_i = u(p_{i_1}, \ldots, p_{i_j}|z_{i_1}, \ldots, z_{i_j})$$

for some u. Again, since X is closed under subterms, it follows from (17) that

$$r'_i = u(p'_{i_1}, \ldots, p'_{i_j}|z_{i_1}, \ldots, z_{i_j}) \in X.$$

But this means that q corresponds to r'_i, so $q \in X^+$. □

Lemma 20 *If $\equiv$ is compositional and X is closed under subterms, then $\equiv^+$ is transitive.*

Proof. Transitivity is an easy consequence of the fact that, when $\equiv$ is compositional and X is closed under subterms,

(18) If s corresponds to both $s' \in X$ and $s'' \in X$, then $s' \equiv s''$.

But (18) in turn follows by a direct application of the Generalized Compositionality Lemma. $\Box$

Before showing that $\equiv^+$ is also compositional under these conditions, let us first note that the whole construction depends on $\equiv$ being non-husserlian. In other words, we have the

Fact 21 *If $\equiv$ is compositional and husserlian, then $\equiv^+ = \equiv$.*

Proof. This follows from Lemma 18 and

$$s \equiv^+ t \;\Rightarrow\; s, t \in X. \tag{19}$$

To prove (19), suppose

$$s = s_0(p_1, \dots, p_k | x_1, \dots, x_k) \text{ and } s' = s_0(p'_1, \dots, p'_k | x_1, \dots, x_k) \in X,$$

where $p_i \equiv p'_i$, $1 \leq i \leq k$. By (k uses of) the Husserl property, it follows that $s \in X$. Similarly, $t \in X$. $\Box$

Lemma 22 *If $\equiv$ is compositional and X is closed under subterms, then $\equiv^+$ is compositional.*

Proof. By Fact 7 in Section 3.7, it is enough to show that $\equiv^+$ is a (partial) congruence relation , i.e., that Congr($\equiv^+$) holds. Thus, suppose

$$p_i \equiv^+ q_i$$

for $1 \leq i \leq k$, and $\alpha(p_1, \dots, p_k), \alpha(q_1, \dots, q_k) \in X^+$. We then have, by definition, for $1 \leq i \leq k$,

$$p_i = s_i(p_{i1}, \dots, p_{ik_i} | x_{i1}, \dots, x_{ik_i}) \tag{20}$$

$$q_i = t_i(q_{i1}, \dots, q_{il_i} | y_{i1}, \dots, y_{il_i}), \tag{21}$$

with

$$s_i(p'_{i1}, \dots, p'_{ik_i} | x_{i1}, \dots, x_{ik_i}) \equiv t_i(q'_{i1}, \dots, q'_{il_i} | y_{i1}, \dots, y_{il_i}), \tag{22}$$

where $p_{ij} \equiv p'_{ij}$, $1 \leq j \leq k_i$, and $q_{ij} \equiv q'_{ij}$, $1 \leq j \leq l_i$. Furthermore,

$$\alpha(p_1, \dots, p_k) = s_0(a_1, \dots, a_m | y_1, \dots, y_m), \tag{23}$$

$$\alpha(q_1, \dots, q_k) = t_0(b_1, \dots, b_n | z_1, \dots, z_n), \tag{24}$$

where

$$s' = s_0(a'_1, \dots, a'_m | y_1, \dots, y_m) \in X, \tag{25}$$

$$t' = t_0(b'_1, \dots, b'_n | z_1, \dots, z_n) \in X, \tag{26}$$

and $a_i \equiv a'_i$, $1 \leq i \leq m$, and $b_j \equiv b'_j$, $1 \leq j \leq n$.

If we can show that

$$s' \equiv t', \tag{27}$$

it follows by the definition of $\equiv^+$ that

$$\alpha(p_1, \dots, p_k) \equiv^+ \alpha(q_1, \dots, q_k),$$

as desired.

Assume in the proof that all the p_i, q_i, p_{ij}, a_r, etc. are distinct *occurrences* of subterms in the respective terms; as with the Generalized Compositionality Lemma this is no loss of generality.

We distinguish the various ways in which (the occurrences) $p_1, \dots, p_k$ may be related to (the occurrences) $a_1, \dots, a_m$, and similarly for $q_1, \dots, q_k$ and $b_1, \dots, b_n$. Call p_i (q_i) *small* if it is a proper subterm of one of $a_1, \dots, a_m$ ($b_1, \dots, b_n$).

Case 1: Some p_i is small. But the only term that p_i can be a proper subterm of is $\alpha(p_1, \dots, p_k)$, so in this case $\alpha(p_1, \dots, p_k) = a_j$ for some j, and *all* the p_i are small.

Subcase 1.1: Some q_i is small. As above, $\alpha(q_1, \dots, q_k) = b_l$ for some l, and all the q_i are small. Thus $\alpha(p_1, \dots, p_k)$ and $\alpha(q_1, \dots, q_k)$ are in X. Also, $p_1, \dots, p_k, q_1, \dots, q_k \in X$ since X is closed under subterms. Therefore, $p_i \equiv q_i$ for $1 \leq i \leq k$, so $\alpha(p_1, \dots, p_k) \equiv \alpha(q_1, \dots, q_k)$ by Comp($\equiv$), and hence $\alpha(p_1, \dots, p_k) \equiv^+ \alpha(q_1, \dots, q_k)$.

Subcase 1.2: No q_i is small. For each i:

> Let $b_{f_i(1)}, \dots, b_{f_i(r_i)}$ be those among the occurrences $b_1, \dots, b_n$ which are subterms of q_i.
>
> That is, $f_i : \{1, \dots, r_i\} \to \{1, \dots, n\}$ enumerates these subterms (without repetitions). Clearly, if $i \neq i'$, then *range*(f_i) is disjoint from *range*$(f_{i'})$. ($r_i = 0$ is allowed; then q_i is disjoint from all the b_j. If $r_i = 1$ then $b_{f_i(1)}$ could be a proper subterm of q_i, or $b_{f_i(1)} = q_i$.)

Now, it follows from our assumptions and (21) that there are terms v_i such that

(28)

$$q_i = v_i(b_{f_i(1)}, \dots, b_{f_i(r_i)} | x_{f_i(1)}, \dots, x_{f_i(r_i)}) = t_i(q_{i1}, \dots, q_{il_i} | y_{i1}, \dots, y_{il_i}).$$

Here $t_i(q'_{i1}, \dots, q'_{il_i}|y_{i1}, \dots, y_{il_i})$ is in X by (22), and if we define

$$q'_i = v_i(b'_{f_i(1)}, \dots, b'_{f_i(r_i)}|x_{f_i(1)}, \dots, x_{f_i(r_i)})$$

for $1 \le i \le k$, then we have, in view of (24) and (26),

$$\alpha(q'_1, \dots, q'_k) = t',$$

so q'_i is a subterm of t', and hence it is also in X. Thus, the Generalized Compositionality Lemma applies to (28), and we get

$$q'_i \equiv t_i(q'_{i1}, \dots, q'_{il_i}|y_{i1}, \dots, y_{il_i}).$$

It follows from ordinary compositionality, using (22), (20), and the fact that $p_i \in X$ in the present subcase, that

$$p_i \equiv q'_i$$

for $1 \le i \le k$. Hence,

$$s' = a'_j \equiv a_j = \alpha(p_1, \dots, p_k) \equiv \alpha(q'_1, \dots, q'_k) = t',$$

again by Comp($\equiv$), since all of these terms are in X.

Case 2: No p_i is small.

Subcase 2.1: Some q_i is small. This case is symmetric to Subcase 1.2.

Subcase 2.2: No q_i is small. Let $b_{f_i(l)}$, v_i, and q'_i be as in Subcase 1.2. Similarly, if we assume that

> $a_{g_i(1)}, \dots, a_{g_i(n_i)}$ are those among the occurrences $a_1, \dots, a_m$ which are subterms of p_i,

there are terms u_i such that

$$p_i = u_i(a_{g_i(1)}, \dots, a_{g_i(n_i)}|x_{g_i(1)}, \dots, x_{g_i(n_i)}) = s_i(p_{i1}, \dots, p_{ik_i}|x_{i1}, \dots, x_{ik_i}),$$

and letting

$$p'_i = u_i(a'_{g_i(1)}, \dots, a'_{g_i(n_i)}|x_{g_i(1)}, \dots, x_{g_i(n_i)}),$$

we conclude as before, using the Generalized Compositionality Lemma, that

$$p'_i \equiv q'_i$$

for $1 \le i \le k$. From this we get

$$s' = \alpha(p'_1, \dots, p'_k) \equiv \alpha(q'_1, \dots, q'_k) = t'$$

by Comp($\equiv$), and the proof of the lemma is finished. □

8.2 Concluding the Proof

Now we can finish the proof of Theorem 12. We are assuming that $\equiv_\mu$ is compositional and that its domain X is closed under subterms. Define

$$\equiv_0 \, = \, \equiv_\mu$$

$$\equiv_{n+1} \, = \, (\equiv_n)^+$$

$$X_n = dom(\equiv_n),$$

and let

$$\equiv^* \, = \, \bigcup_{n<\omega} \equiv_n$$

$$Y = dom(\equiv^*).$$

From the results in the previous subsection we see that each $\equiv_{n+1}$ is a compositional synonymy which extends $\equiv_n$ and whose domain X_{n+1} is closed under subterms. It follows immediately that $\equiv^*$ is a compositional synonymy whose domain is closed under subterms. Also, we see (by induction) that

if $s, t \in X_0$, then $(s \equiv_{n+1} t \Rightarrow s \equiv_n t)$.

Hence, $\equiv^*$ extends $\equiv_\mu$. (Note that this uses the compositionality of each $\equiv_{n+1}$; cf. Lemma 18). Finally, by the definition of $\equiv_{n+1}$,

(29) if $p_i \equiv_n q_i$ for $1 \leq i \leq k$, $s(p_1, \ldots, p_k | x_1, \ldots, x_k) \in X_n$ and the term $s(q_1, \ldots, q_k | x_1, \ldots, x_k)$ is grammatical, then $s(q_1, \ldots, q_k | x_1, \ldots, x_k) \in X_{n+1}$.

But this means that $\equiv^*$ satisfies the following weaker version of the Husserl property:

(30) Suppose that $p_i \equiv^* q_i$ for $1 \leq i \leq k$, that $s(p_1, \ldots, p_k | x_1, \ldots, x_k) \in Y$, and that $s(q_1, \ldots, q_k | x_1, \ldots, x_k)$ is grammatical. Then $s(q_1, \ldots, q_k | x_1, \ldots, x_k) \in Y$.

For take n so that $p_i \equiv_n q_i$ for $1 \leq i \leq k$ and $s(p_1, \ldots, p_k | x_1, \ldots, x_k) \in X_n$. By (29), $s(q_1, \ldots, q_k | x_1, \ldots, x_k) \in X_{n+1} \subseteq Y$.

Now, let $\equiv^1$ be the total one-point extension of $\equiv^*$, as defined in the proof of Corollary 11 (Section 5). It is clear that the argument in that proof showing that the one-point extension is compositional goes through when (30) replaces the assumption of the Husserl property. Thus, $\equiv^1$ is indeed a total compositional extension of $\equiv_\mu$, and the proof is complete. $\square$

9 Remarks and Questions

9.1 Uniqueness

Theorem 12 concerns the existence of a total compositional extension of $\equiv_\mu$. In general, there may be several distinct such extensions, among them:

(31) $\quad \equiv^{\min} = \bigcap\{\equiv\, \supseteq\, \equiv_\mu\colon \equiv \text{ is a total compositional synonymy for } \mathbf{E}\}.$

$\equiv^{\min}$ is *always* a total compositional synonymy for $\mathbf{E}$, without any assumptions at all on $\equiv_\mu$. (As pointed out in Hodges [9], this only depends on the form of the condition defining $\equiv^{\min}$.) But without such assumptions $\equiv^{\min}$ need not *extend* $\equiv_\mu$. For example, if μ is the semantics in Example 8 (Section 3.8), we will have $\beta(\alpha(a)) \equiv^{\min} \beta(c)$, though $\beta(\alpha(a))$ and $\beta(c)$ are both in $dom(\mu)$, and $\beta(\alpha(a)) \not\equiv_\mu \beta(c)$.

But (again pointed out by Hodges), as soon as we know that there *exists* a total compositional synonymy, say $\equiv'$, which does extend – or even refine – $\equiv_\mu$, it follows that $\equiv^{\min}$ also extends $\equiv_\mu$: If $p, q \in dom(\mu)$ and $p \equiv^{\min} q$, then, since $\equiv^{\min} \subseteq\, \equiv'$, $p \equiv' q$, and so, since $\equiv'$ refines $\equiv_\mu$, $p \equiv_\mu q$. Therefore, Theorem 12 implies that if μ is a compositional semantics whose domain is closed under subterms, then $\equiv^{\min}$ is uniquely defined as the smallest total compositional extension of $\equiv_\mu$.

However, Hodges also shows that the $\subseteq$-smallest total compositional extension need not be the most interesting one. For example, in the cofinal and husserlian case it is not in general equivalent to the fregean extension. In general, it looks like an interesting task to find criteria to choose between the various compositional extensions (or refinements) of a given semantics, criteria which apply also under the very general circumstances assumed in Theorem 12, for example.

9.2 Another Route to Theorem 12

After reading the penultimate draft of this paper, Tim Fernando found another proof of the main result. That proof is in many ways illuminating, and I will outline it here (with his permission). The idea is to prove directly the fact mentioned about $\equiv^{\min}$ above. For this, one needs to analyze the construction of $\equiv^{\min}$ more carefully. The first part of this analysis does not depend on the fact that we are dealing with terms in a term algebra.

Let $\equiv_\mu$ be a partial synonymy with domain $X \subseteq B$, but for the moment assume no particular structure of the elements of B, just that B is the domain of some partial algebra with signature Σ. Recall that the condition Congr (Section 3.7) is still applicable in this case. Now do the following:

Step 1: Let $\equiv_{\mu,B}$ be the union of $\equiv_\mu$ and the identity relation on B. ($\equiv_{\mu,B}$ is the smallest total synonymy extending $\equiv_\mu$.)

Step 2: Enlarge $\equiv_{\mu,B}$ inductively as follows: at each step,

- if (p, q) and (q, r) are in the relation obtained so far, add (p, r);
- if (p_i, q_i) are in the relation obtained so far for $1 \leq i \leq n$, and if $\alpha \in \Sigma$ and $\alpha(p_1, \dots, p_n), \alpha(q_1, \dots, q_n)$ are defined, add $(\alpha(p_1, \dots, p_n), \alpha(q_1, \dots, q_n))$.

Let $\equiv^{\Sigma}_{\mu,B}$ be the union of all these relations.

Lemma 23 (Fernando)

(i) $\equiv^{\Sigma}_{\mu,B}$ *is included in every total synonymy R s.t. Congr(R) and $\equiv_\mu \subseteq R$.*

(ii) $\equiv^{\Sigma}_{\mu,B}$ *is a total synonymy and Congr$(\equiv^{\Sigma}_{\mu,B})$ holds.*

(iii) *There is a total synonymy satisfying Congr which extends $\equiv_\mu$ iff $\equiv^{\Sigma}_{\mu,B}$ refines $\equiv_\mu$.*

The proof is not hard. For (i), one checks that the relation obtained at each step of the inductive construction includes such an R. For (ii), only transitivity and Congr need to be verified, and this too is done following the construction of $\equiv^{\Sigma}_{\mu,B}$. (iii) follows from (i) and (ii).

This construction, which is the inductive analogue of a co-inductive construction used in Fernando [2], is somewhat similar to the one in Section 8. But it is more direct and more general, adding (after the first step) just what is clearly needed to obtain Congr while maintaining transitivity, whereas the construction in 8.1 looks into the structure of terms (and insures Congr and transitivity at the same time).

Now let $B = GT$ as before and assume that X is closed under subterms. By Fact 7 (Section 3.7), Congr then amounts to real compositionality. Using this, and Fernando's lemma above, it follows that

$$\equiv^{\min} = \equiv^{\Sigma}_{\mu,GT} \, . \qquad (32)$$

To prove Theorem 12, it suffices by the lemma to prove

Lemma 24 (Fernando) *If Congr$(\equiv_\mu)$ and X is closed under subterms, then $\equiv^{\Sigma}_{\mu,GT}$ refines $\equiv_\mu$.*

This may look straightforward but isn't quite, or so it seems. Suppose (p, q) has been added at a certain stage in the construction of $\equiv^{\Sigma}_{\mu,GT}$, and $p, q \in X$. We want to conclude that $p \equiv_\mu q$. The problem comes with transitivity: (p, q) may have been added because (p, b) and (b, q) were already there, but we do not know that $b \in X$, and therefore cannot straightforwardly apply an inductive hypothesis to reach the desired conclusion. Fernando's proof has the look of a cut-elimination argument, viewing the addition of pairs to $\equiv^{\Sigma}_{\mu,GT}$ as derivations in a formal system, and expanding out final applications of transitivity in an intricate manner.[10]

[10]Though the complexities in the two proofs of Theorem 12 are perhaps comparable, Fernando's proof has the advantage of relying on a general fact about extending equivalence relations to congruence relations, and of applying techniques familiar from proof theory when a detailed look at the structure of terms in the partial term algebra is needed.

One may ask if $\equiv^{\Sigma}_{\mu,GT}$ is the same relation as $\equiv^1$ in Section 8.2, but that will not in general be the case, since Fernando's construction starts with the very *fine* addition of the identity relation and then inductively adds just what is needed, whereas $\equiv^*$ is obtained by first inductively performing the step in 8.1 and then letting the very *coarse* one-point extension take care of what's left.

9.3 An Open Problem

There is one main case that the results mentioned here do not settle, namely, when $dom(\mu)$ is cofinal in GT but the Husserl property fails.

> **Open Problem**: For a cofinal semantics, what is the condition besides compositionality which (in the absence of the Husserl property) guarantees that it has a total compositional extension?

Recall from Hodges' Theorem that in this case too, the fregean cover is compositional, but it does not extend the given semantics. Among the variants and generalizations of that theorem, this problem may well be the hardest one.

9.4 Many-Sorted Algebra

In a sense the partial algebra approach to formal semantics generalizes the approach via many-sorted algebras (cf. Section 2), since it is rather clear that each many-sorted algebra

$$\mathbf{A} = \langle (A_s)_{s\in S}, (F_\gamma)_{\gamma\in\Gamma}\rangle$$

can be turned into a one-sorted partial algebra $\mathbf{E}_A$ with domain $E_A = \bigcup_{s\in S} A_s$, whereas on the other hand not every partial algebra corresponds to a many-sorted one.

In more detail: Given $\mathbf{A}$, if F_γ is an operation with domain $A_{s_1}\times\cdots\times A_{s_n}$, define a partial operation $\underline{\gamma}$ from $(E_A)^n$ to E_A by

$$\underline{\gamma}(e_1,\ldots,e_n) = \begin{cases} F_\gamma(e_1,\ldots,e_n), & \text{if } e_i\in A_{s_i}, 1\le i\le n \\ \text{undefined}, & \text{otherwise.}\end{cases}$$

If A_0 is the set of atoms in $\mathbf{A}$, we let $\mathbf{E}_A = \langle E_A, A_0, \underline{\gamma}\rangle_{\gamma\in\Gamma}$ (or, with the notation used in Section 3.1 to emphasize the fact that the algebra is generated, $\mathbf{E}_A = \langle [A_0], \underline{\gamma}\rangle_{\gamma\in\Gamma}$). Then $\mathbf{E}_A$ corresponds to $\mathbf{A}$ in the sense that, if for strings t of the form '$F_\gamma(t_1,\ldots,t_n)$' we define (inductively) $t^* = \gamma(t_1^*,\ldots,t_n^*)$, whereas $a^* = a$ when a is an atom, we have the following

Fact 25 *$t\in T(\mathbf{A})$ iff $t^*\in GT(E_A)$.*

(Recall that $T(\mathbf{A})$ is the term algebra of $\mathbf{A}$.) However, the syntactic categories of $\mathbf{E}_A$ need not correspond to the sorts of $\mathbf{A}$. Suppose $a\in A_s\cap A_{s'}$, $b\in A_s - A_{s'}$, and that F_γ is a unary operation with domain $A_{s'}$. Then $a \not\sim_{GT(E_A)} b$ (since $\gamma(a)$ but not $\gamma(b)$ is grammatical) even though a and b have the common sort s.

The possibility for an object to belong to more than one sort is, however, the only obstacle:

Fact 26 *If* $(A_s)_{s \in S}$ *is a partition of* E_A*, then its corresponding equivalence relation is* $\sim_{GT(E_A)}$.

Hendriks [4] (p. 141, footnote 10) argues that the antecedent in the above fact is a natural requirement on many-sorted algebras that represent the syntax of natural languages.

Thus, we can easily turn a many-sorted algebra into a one-sorted partial algebra, and if the sorts partition the expressions, the sorts can be recovered in the partial algebra, even though they are not primitive there. But there is no equally obvious way to go in the opposite direction. This is because the domain of an operation in a many-sorted algebra must be a cartesian product, but in a partial algebra $\mathbf{E}$ there may well be an operation $\underline{\alpha}$ such that, say, $(a, b), (a', b') \in dom(\underline{\alpha})$, but $(a, b') \notin dom(\underline{\alpha})$.

A natural question is if the extension theorems discussed here (Hodges' Theorem and its variants) can be transferred to the many-sorted approach.

9.5 Syntactic Categories

Continuing the discussion above, the crucial issue seems to be which role (syntactic) categories are supposed to play. If they are to exactly match intersubstitutability, we have seen that they can in principle be dispensed with (i.e., defined) in Hodges' partial algebra approach, and likewise in the many-sorted Montague approach, provided Hendriks' requirement above is satisfied.

However, the linguistic intuitions behind the assignment of categories to expressions presumably go far beyond considerations about substitution, and may even clash with such considerations. I shall not pursue the matter here, but merely note that it would be interesting to see how other notions of syntactic category could be integrated with Hodges' abstract approach to grammar. Recent work by Keenan and Stabler [12] is particularly relevant here. They develop an abstract framework for grammars – with the goal, among others, to apply a precise notion of syntactic (or, more generally, linguistic) *invariant* to existing grammars – and they too construe grammars as partial algebras, the main difference from Hodges' approach being that primitive syntactic categories are built into their notion of an expression. But, unless extra axioms are added, these categories need not be related in any simple way to substitution.[11]

[11] It can be quite natural *not* to have the domains of grammar rules being cartesian products. Borrowing an example from Keenan and Stabler [12] (cf. their grammar Kor), suppose a case marking rule combines expressions of category NP with expressions of category Case (say, *-nom, -acc, -dat,* ...). Not all NP's need to combine with all cases; for example, a reflexive pronoun cannot combine with *-nom*. As noted, this could be a problem for a total many-sorted approach.

References

[1] Cameron, P. and Hodges, W.: Some combinatorics of imperfect information, *Journal of Symbolic Logic* **66** (2001), 673–864.

[2] Fernando, T.: Ambiguity under changing contexts, *Linguistics and Philosophy* **20** (1997), 575–606.

[3] Grätzer, G.: *Universal Algebra*, 2nd ed. Springer-Verlag, New York, 1968.

[4] Hendriks, H.: *Studied Flexibility. Categories and Types in Syntax and Semantics*, ILLC Dissertation Series 1993–5, ILLC, University of Amsterdam, 1993.

[5] Hendriks, H.: Compositionality and model-theoretic interpretation, *Journal of Logic, Language and Information* **10** (2001), 29–48.

[6] Hintikka, J. and Sandu, G: Game-theoretical semantics, in A. ter Meulen and J. van Benthem (eds.), *Handbook of Logic and Language*, Elsevier, 1997, 361–410.

[7] Hodges, W.: Compositional semantics for a language of imperfect information, *Journal of the IPGL* **5** (1997), 539–563.

[8] Hodges, W.: Some strange quantifiers, in J. Mycielski et al. (eds.), *Structures in Logic and Computer Science*, LNCS 1261, Springer, Berlin, 1997, 51–65.

[9] Hodges, W.: Formal features of compositionality, *Journal of Logic, Language and Information* **10** (2001), 7–28.

[10] Hodges, W.: Composition of meaning. A class at Düsseldorf, unpublished lecture notes, 2003.

[11] Janssen, T.: *Foundations and Applications of Montague Grammar*, CWI Tracts 19 and 28, Amsterdam, 1986.

[12] Keenan, E. and Stabler, E.: Linguistic invariants and language variation, 2003, forthcoming in P. Hajek, L. Valdés, and D. Westerståhl (eds), *Proceedings of the 12th International Congress of Logic, Methodology and Philosophy of Science*, Elsevier, Amsterdam.

[13] Kracht, M.: Strict compositionality and literal movement grammars, in *Logical Aspects of Computational Linguistics '98*, LNAI 2014, Springer, Berlin, 2001.

[14] Montague, R.: Universal Grammar, *Theoria* **36** (1970). Also in R. Thomason (ed.), *Formal Philosophy*, Yale UP, New Haven, 1974, 373–398.

[15] Nunberg, R., Sag, I., and Wasow, T.: Idioms, *Language* **70** (1974), 491–538.

[16] Sandu, G. and Hintikka, J.: Aspects of compositionality, *Journal of Logic, Language and Information* **10** (2001), 49–61.

[17] Westerståhl, D.: On mathematical proofs of the vacuity of compositionality. *Linguistics and Philosophy* **21** (1998), 635–643.

[18] Westerståhl, D.: On the compositionality of idioms; an abstract approach, in D. Barker-Plummer, D. Beaver, J. van Benthem, and P. Scotto di Luzio (eds.), *Words, Proofs, and Diagrams*, CSLI Publications, Stanford, 2002, 241–271.

[19] Zadrozny, W.: From compositional to systematic semantics, *Linguistics and Philosophy* **17** (1994), 329–342.