# Self-Commuting Quantifiers

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**Abstract**

We characterize the generalized quantifiers $Q$ which satisfy the scheme $QxQy\phi \leftrightarrow QyQx\phi$, the so-called self-commuting quantifiers, or quantifiers with the Fubini property.

## 1 Introduction

Let $Q$ be a type $\langle 1 \rangle$ generalized quantifier on a domain $M$, $Q \subseteq \mathcal{P}(M)$. $Q$ is *self-commuting* (on $M$) if the scheme

$$QxQy\phi \leftrightarrow QyQx\phi$$

is valid. Put differently, $Q$ is self-commuting if, for all $R \subseteq M^2$,

(1) $$QQR \Leftrightarrow QQR^{-1},$$

where, for type $\langle 1 \rangle$ quantifiers $Q_1, Q_2$ on $M$,

(2) $$Q_1Q_2R \Leftrightarrow Q_1\{a \in M : Q_2R_a\}$$

($R_a = \{b \in M : Rab\}$).

Familar examples are $\forall = \{M\}$ and $\exists = \{A \subseteq M : A \neq \emptyset\}$, as well as, for $a \in M$, $F_a = \{A \subseteq M : a \in A\}$. The quantifiers $F_a$, which we will call

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the *atoms* on $M$, have the stronger property of *scopelessness*: $Q$ is *scopeless* if for all $Q'$ and all $R \subseteq M^2$,

$$QQ'R \Leftrightarrow Q'QR^{-1}$$

Zimmermann has shown (cf. [5]) that the scopeless quantifiers on $M$ are precisely the atoms. In Montague style semantics for natural language, atoms serve as interpretations of *proper names*— $[\![John]\!] = F_{\mathrm{John}}$— which indeed lack scope with respect to quantified phrases: *John saw most of the films* means the same as *Most of the films were seen by John*, but the equivalence fails if *John* is replaced by, say, *most students*.

If $Q$ is *almost all* in the measure-theoretic sense, so that $Q$ consists of the subsets of $M$ of measure 1, then self-commutativity is reminiscent of Fubini's theorem, by which (1) holds for all *measurable* $R$. By this analogy, self-commutativity has been called the *Fubini property*. To require that (1) holds for *all* $R$ is much stronger, and van Lambalgen shows in [2] that under AC (which we assume here) there can be no self-commutative non-principal filters on $M$. The present characterization of self-commutativity is a generalization of this result.

In the next section we give more examples of self-commuting quantifiers. The characterization of self-commutativity is given in section 3, together with some corollaries. Section 4 is devoted to the main part of the proof of the result, and section 5 considers generalizations.

# 2 Examples

**Fact 2.1** *All unions and intersections of atoms are self-commuting.*

*Proof*: We have, for $B \subseteq M$,

$$(\bigcup_{a\in B} F_a)(\bigcup_{a\in B} F_a)R \Longleftrightarrow \bigvee_{a\in B}(\bigvee_{b\in B} Rab) \Longleftrightarrow \bigvee_{a,b\in B} Rab \Longleftrightarrow \bigvee_{a,b\in B} R^{-1}ab$$

and similarly for intersection. □

This includes the *trivial* quantifiers $\mathbf{0} = \emptyset = \bigcup\emptyset$ and $\mathbf{1} = \mathcal{P}(M) = \bigcap\emptyset$. Among the unions of atoms we find interpretations of disjunctive noun phrases like *John or Bill or Mary* ($F_{\mathrm{John}} \cup F_{\mathrm{Bill}} \cup F_{\mathrm{Mary}}$), and noun phrases like *some books* ($\{A \subseteq M : A \cap B \neq \emptyset\} = \bigcup_{a \in B} F_a$, where $B$ is the set of books), as well as

$$\exists = \bigcup_{a \in M} F_a$$

The intersections of atoms are precisely the *principal filters* on $M$,

$$F_B = \{A \subseteq M : B \subseteq A\} = \bigcap_{a \in B} F_a$$

such as the interpretation of *John and Bill and Mary*, or *the men* ($\bigcap_{a \in B} F_a$, where $B$ is the set of men), or *John's ten bikes* (when $B$ is the set of bikes that John owns: $\bigcap_{a \in B} F_a$ if $|B| = 10$, and $\mathbf{0}$ otherwise). Also, $\forall = F_M$.

The *outer* and *inner negation*, and the *dual* of $Q$ are defined as usual:

$$\neg Q = \mathcal{P}(M) - Q$$

$$Q\neg = \{A \subseteq M : M - A \in Q\}$$

$$Q^d = \neg(Q\neg) = (\neg Q)\neg$$

$Q$ is *self-dual* if $Q^d = Q$ (so the self-dual filters are just the ultrafilters).

**Fact 2.2** *If $Q$ is self-commuting, so is $Q^d$.*

*Proof*: By elementary quantifier manipulations: $Q^d Q^d R \Leftrightarrow \neg Q \neg \neg Q \neg R \Leftrightarrow \neg Q Q \neg R \Leftrightarrow \neg Q Q \neg R^{-1}$(since $Q$ is self-commuting) $\Leftrightarrow Q^d Q^d R^{-1}$. $\square$

**Lemma 2.3** $\bigcup_{a \in B} F_a$ ($\bigcap_{a \in B} F_a$) *is self-dual iff* $|B| = 1$.

*Proof*: Each $F_a$ is clearly self-dual. For $B = \emptyset$, note that $\emptyset$ and $\mathcal{P}(M)$ are not self-dual: $\emptyset^d = \mathcal{P}(M)$. If $|B| > 1$ then for each $b \in B$, $\{b\} \in \bigcup_{a \in B} F_a - (\bigcup_{a \in B} F_a)^d$, and $M - \{b\} \in (\bigcap_{a \in B} F_a)^d - \bigcap_{a \in B} F_a$. $\square$

If $Q$ is both self-commuting and self-dual, then $\neg Q$ is also self-commuting ($\neg Q \neg Q = Q^d Q = QQ$). So each $\neg F_a$ ($= F_a \neg =$ the principal ideal generated by $a$) is self-commuting. This is an example of a *downward monotone* self-commuting quantifier—note that the quantifiers from Fact 2.1 are all *upward* monotone. In fact, we will see that it is the *only* example. In particular, when $|B| > 1$, $\neg(\bigcup_{a \in B} F_a)$ and $\neg(\bigcap_{a \in B} F_a)$ are not self-commuting.

Our next examples of self-commuting quantifiers are neither upward nor downward monotone.

The *symmetric difference* operation on two sets,

$$A \oplus B = (A - B) \cup (B - A)$$

is (commutative and) associative, so the notation

$$\bigoplus_{i=1}^{n} B_i$$

makes sense (for $n \geq 1$; if $n = 1$, $\bigoplus_{i=1}^{n} B_i = B_1$).

**Fact 2.4** *$a \in \bigoplus_{i=1}^{n} B_i \iff |\{i : a \in B_i\}|$ is odd.*

Now consider quantifiers $\bigoplus_{i=1}^{n} F_{a_i}$. From Fact 2.4 we see that

$$X \in \bigoplus_{i=1}^{n} F_{a_i} \iff |X \cap \{a_1, \ldots, a_n\}| \text{ is odd} \tag{3}$$

**Fact 2.5** *$\bigoplus_{i=1}^{n} F_{a_i}$ is self-commuting.*

*Proof*: Let $Q = \bigoplus_{i=1}^{n} F_{a_i}$ and $A = \{a_1, \ldots, a_n\}$.

$$\begin{array}{rcl} QQR & \iff & Q\{a : |R_a \cap A| \text{ is odd}\} \\ & \iff & |\{a \in A : |R_a \cap A| \text{ is odd}\}| \text{ is odd} \\ & \iff & |R \cap (A \times A)| \text{ is odd} \end{array}$$

The last equivalence is seen to hold since

$$|R \cap (A \times A)| = |R_{a_1} \cap A| + \ldots + |R_{a_n} \cap A|$$

It is now clear that $QQR \Leftrightarrow QQR^{-1}$. □

An example from natural language might be *John or Mary but not both* ($F_{\text{John}} \oplus F_{\text{Mary}}$). [1]

The following is an easy consequence of Fact 2.4.

[1] Or *either John or Mary*, if one accepts (but this is doubtful) a reading of this with exclusive disjunction. Then, an instance of the self-commutativity of this quantifier is the perhaps not immediately obvious equivalence of the two sentences *Either John or Mary criticized either John or Mary* and *Either John or Mary was criticized by either John or Mary.* Note that the equivalence continues to hold if one of the two noun phrases is replaced by *either Bill or Sue*; this generalization is taken up in section 5. But note also that phrases with three or more disjuncts, say *either John or Mary or Sue*, do not have interpretations as exclusive disjunctions (symmetric differences).

**Fact 2.6** *Let* $Q = \bigoplus_{i=1}^{n} F_{a_i}$ *and* $A = \{a_1, \ldots, a_n\}$*. If* $|A|$ *is odd, then* $Q\neg = \neg Q$ *and* $Q^d = Q$*. If* $|A|$ *is even, then* $Q\neg = Q$ *and* $Q^d = \neg Q$*.*

This gives us our last examples of self-commuting quantifiers.

**Corollary 2.7** $\neg \bigoplus_{i=1}^{n} F_{a_i}$ *is self-commuting.*

*Proof*: With $Q = \bigoplus_{i=1}^{n} F_{a_i}$ and $A = \{a_1, \ldots, a_n\}$ we have: If $|A|$ is odd, then $\neg Q \neg Q = Q^d Q = QQ$, and if $|A|$ is even, then $\neg Q \neg Q = Q^d Q^d$, so the result follows from Facts 2.2 and 2.5. □

# 3 Characterization

The main result of this paper shows that the examples of self-commuting quantifiers in the previous section are in fact the only ones.

**Theorem 3.1** *$Q$ is self-commuting iff $Q$ is either a union or an intersection of atoms, or a finite symmetric difference of atoms, or a negation of such a symmetric difference.*

We now derive this theorem from the following Main Lemma, which is proved in the next section.

**Lemma 3.2 (Main Lemma)** *If $Q$ is self-commuting and $\emptyset \notin Q$, then $Q$ is either a union or an intersection of atoms, or a finite symmetric difference of atoms.*

*Proof of Theorem 3.1*: 'If': This follows from the results in section 2.
'Only if': If $\emptyset \notin Q$ we can use the Main Lemma, so suppose $\emptyset \in Q$. Assume first that $M \in Q$. Then $\emptyset \notin Q^d$, so the Main Lemma applied to $Q^d$ gives

$$\begin{array}{rcl} Q^d & = & \bigcup_{a \in B} F_a \\ \text{or} \quad Q^d & = & \bigcap_{a \in B} F_a \\ \text{or} \quad Q^d & = & \bigoplus_{i=1}^{n} F_{a_i} \end{array}$$

It follows that

$$\begin{array}{rrcl} & Q & = & (\bigcup_{a\in B} F_a)^d & = & \bigcap_{a\in B}(F_a)^d & = & \bigcap_{a\in B} F_a \\ \text{or} & Q & = & (\bigcap_{a\in B} F_a)^d & = & \bigcup_{a\in B}(F_a)^d & = & \bigcup_{a\in B} F_a \\ \text{or} & Q & = & (\bigoplus_{i=1}^n F_{a_i})^d \end{array}$$

and, by Fact 2.6, in the latter case $Q$ is either $\bigoplus_{i=1}^n F_{a_i}$ or $\neg\bigoplus_{i=1}^n F_{a_i}$.

Now suppose that $M \notin Q$.

CLAIM: $Q$ is self-dual.

*Proof of Claim*: Suppose, for contradiction, that there is a set $A$ such that either $A \in Q$ and $M - A \in Q$, or $A \notin Q$ and $M - A \notin Q$. By assumption, $A \neq \emptyset$ and $A \neq M$. Define the relation $R$ by the following condition:

$$R_a = \begin{cases} \emptyset & \text{if } a \in A \\ M & \text{if } a \in M - A \end{cases}$$

If $A, M - A \in Q$, then $\{a : QR_a\} = A \in Q$, but $\{a : Q(R^{-1})_a\} = M \notin Q$. This contradicts the self-commutativity of $Q$. If $A, M - A \notin Q$, then $\{a : QR_a\} = A \notin Q$, but $\{a : Q(R^{-1})_a\} = \emptyset \in Q$, again contradicting self-commutativity. This proves the Claim.

Now since $Q$ is self-commuting and self-dual, $\neg Q$ is also self-commuting. Since $\emptyset \notin \neg Q$ we can apply the Main Lemma:

$$\begin{array}{rrcl} & \neg Q & = & \bigcup_{a\in B} F_a \\ \text{or} & \neg Q & = & \bigcap_{a\in B} F_a \\ \text{or} & \neg Q & = & \bigoplus_{i=1}^n F_{a_i} \end{array}$$

But $\neg Q$ is also self-dual, so it follows from Lemma 2.3 that in the first two cases, $|B| = 1$, i.e., $Q = \neg F_a$ for some $a \in M$. And in the third case, $Q = \neg\bigoplus_{i=1}^n F_{a_i}$. □

The theorem generalizes results about self-commutativity in van Benthem [1] and van Lambalgen [2]. Indeed, van Lambalgen's result is used in the proof of the Main Lemma.

We now consider a few corollaries to the theorem. First note the asymmetry between upward and downward monotone self-commutative quantifiers: the unions and intersections of atoms are upward monotone, but

**Corollary 3.3** *The only non-trivial and downward monotone self-commuting quantifiers are the principal ideals $\neg F_a$ for $a \in M$.*

As the proof above shows, the reason is that when $\emptyset \in Q \cap Q^d$, self-commutativity forces $Q$ to be self-dual.

Call $Q$ ISOM if it is a quantifier in the logical sense, i.e., if $(M, A) \cong (M', A')$ and $A \in Q_M$ implies $A' \in Q_{M'}$ (where $Q$ is now a functional relation assigning to each domain $M$ a quantifier $Q_M$ on $M$). Also, let $Q_{odd}$ be the quantifier defined by $A \in Q_{odd}$ iff $|A|$ is odd, and similarly for $Q_{even}$.

**Corollary 3.4** *The only ISOM and self-commuting quantifiers, except the trivial ones, are $\forall$ and $\exists$, and, on finite domains, $Q_{odd}$ and $Q_{even}$.* [2]

*Proof*: If $Q$ is $\bigcup_{a\in B} F_a$ or $\bigcap_{a\in B} F_a$, it is rather clear that $Q$ can only be ISOM if $B$ is either $\emptyset$ or $M$. The first case gives the trivial quantifiers, and the second $\forall$ and $\exists$. Similarly, $\bigoplus_{i=1}^n F_{a_i}$ and $\neg \bigoplus_{i=1}^n F_{a_i}$ can only be ISOM if $\{a_1, \ldots, a_n\} = M$, and then

$$X \in \bigoplus_{a\in M} F_a \Longleftrightarrow |X| \text{ is odd}$$

(Note, by the way, that $Q_{odd}xQ_{odd}yRxy$ always says that $|R|$ is odd, whereas $Q_{even}xQ_{even}yRxy$ says that $|R|$ is odd if $|M|$ is odd, and that $|R|$ is even if $|M|$ is even.) $\Box$

If $Q_1$ and $Q_2$ are type $\langle 1\rangle$ quantifiers, the quantifier $Q_1Q_2$ is of type $\langle 2\rangle$, i.e., it is (on $M$) a set of binary relations on $M$. Call a type $\langle 2\rangle$ quantifier *convertible* if whenever $R$ belongs to it, so does $R^{-1}$. The next corollary generalizes a result in [4].

**Corollary 3.5** *Let $Q = Q_1Q_2$, where $Q$ is non-trivial and $\emptyset \notin Q_2$. $Q$ is convertible iff $Q_1 = Q_2$ or $\neg Q_1 = Q_2$, where $Q_2$ in both cases is as in the Main Lemma.*

[2] Theorem 3.1 in fact stems from my attempt to explain what $Q_{odd}$ and $Q_{even}$ were doing in this result, which was proved in [4] for finite universes. As it turned out, neither ISOM nor the restriction to finite models is needed in the general result.

*Proof*: Observe that the requirement $\emptyset \notin Q_2$ is not really a restriction, since $Q_1Q_2 = Q_1\neg\neg Q_2$. Assume first that $\emptyset \notin Q_1$. It is shown in [4] (using Keenan's so-called Prefix and Product Theorems) how the convertibility of $Q$ then implies that $Q_1 = Q_2$. But then convertibility reduces to self-commutativity, and so the Main Lemma applies. If instead $\emptyset \in Q_1$, we apply the same argument to $\neg Q$. $\square$

The corollary gives a complete characterization of the quantifier pairs $(Q_1, Q_2)$ which satisfy, for all $R \subseteq M^2$,

(4) $$Q_1Q_2R \Leftrightarrow Q_1Q_2R^{-1},$$

a condition which in a natural way generalizes the self-commutativity condition (1). Another obvious generalization of (1) to pairs of quantifiers is

(5) $$Q_1Q_2R \Leftrightarrow Q_2Q_1R^{-1}$$

We will consider this condition in section 5.

# 4 Proof of the Main Lemma

The proof of the Main Lemma to be given below is a substantial simplification, due to Lauri Hella, of my considerably more involved original proof, and is presented here with his permission.

Assume, for this section, that $Q$ is self-commutative and that $\emptyset \notin Q$. We first present four preliminary lemmas. The proof of the first lemma is in fact (as Hella pointed out) a standard argument (due to Sierpinski) from measure theory that there can be no Borel well-ordering of the reals. This uses the fact that Borel relations are Lebesgue measurable, and the Fubini theorem. But the argument does not use any other properties of the reals, and so goes through in the present abstract setting.

**Definition 4.1** $Q$ is *splittable* if $A \in Q$ and $B \subseteq A$ implies that either $B \in Q$ or $A - B \in Q$.

For example, $\bigcup_{a\in B} F_a$ and $\bigoplus_{i=1}^{n} F_{a_i}$ are splittable, but not $\bigcap_{a\in B} F_a$ (when $|B| > 1$).

**Lemma 4.1** *If $Q$ is splittable, then $A \in Q$ implies $\{a\} \in Q$ for some $a \in A$.*

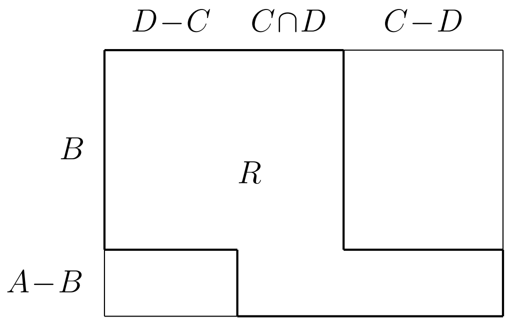
*Proof*: Clearly it is enough to prove that for infinite $A$, $A \in Q$ implies $B \in Q$ for some $B \subseteq A$ with $|B| < |A|$. Suppose this fails for some infinite $A \in Q$. Let $R$ be a well-ordering of $A$ with order type $|A|$, so that all proper initial segments have cardinality $< |A|$. Thus, $B \notin Q$ for all proper initial segments $B$. By splittability, $C \in Q$ for all proper end segments $C$. But this means that $QQR$ is true but $QQR^{-1}$ is false, contradicting self-commutativity. $\square$

**Lemma 4.2** *If $Q$ is not splittable, it is closed under finite intersections.*

*Proof*: Suppose there are $A \in Q$, $B \subseteq A$, such that $B \notin Q$ and $A-B \notin Q$. If $C, D \in Q$, define $R$ as follows:

$$R_a = \begin{cases} C & \text{if } a \in A - B \\ D & \text{if } a \in B \end{cases}$$

Here and in what follows, this is taken to mean that $R_a = \emptyset$ for all $a$ not explicitly mentioned in the defining condition. So $R \subseteq A \times (C \cup D)$, and we can draw a simple picture of $R$ as follows:




Then $\{a : QR_a\} = A \in Q$. Also, we see (looking at the figure) that $\{a : Q(R^{-1})_a\} = C \cap D$. Thus, by self-commutativity, $C \cap D \in Q$. (Similar pictures are helpful for some of the proofs below.) $\square$

The next lemma says that $Q$ must be 'almost' upward monotone.

**Lemma 4.3** *If $B \in Q$, $B \subseteq A$, and $A - B \notin Q$, then $A \in Q$.*

*Proof*: Assume, for contradiction, that $B \in Q$, $B \subseteq A$, and $A - B \notin Q$, but $A \notin Q$. Define $R$ by

$$R_a = \begin{cases} A - B & \text{if } a \in A - B \\ A & \text{if } a \in B \end{cases}$$

Then $\{a : QR_a\} = \emptyset \notin Q$. On the other hand, $\{a : Q(R^{-1})_a\} = B \in Q$, which contradicts the fact that $Q$ is self-commuting. □

The final lemma says roughly that if $Q$ is not upward monotone, it must be 'alternating'.

**Lemma 4.4** *Suppose $Q$ is not upward monotone. Then $B \in Q$, $B \subseteq A$, and $A - B \in Q$ implies that $A \notin Q$.*

*Proof*: If $Q$ is not upward monotone, it follows from Lemma 4.3 that there are $C, D$ such that $C \in Q$, $C \subseteq D$, $D \notin Q$, and $D - C \in Q$. Take any $A, B$ with $B \in Q$, $B \subseteq A$, and $A - B \in Q$. Let $R$ be defined by

$$R_a = \begin{cases} B & \text{if } a \in C \\ A - B & \text{if } a \in D - C \end{cases}$$

Then $\{a : QR_a\} = D \notin Q$. But $\{a : Q(R)_a^{-1}\} = A$, so, by self-commutativity, $A \notin Q$. □

We can now start proving the Main Lemma.

*Case 1*: $Q$ is closed under finite intersections.

By Lemma 4.3, and since $\emptyset \notin Q$, it then follows that $Q$ is upward monotone. Thus, $Q$ is a filter. It now follows from a result in [2] that $Q$ is closed under arbitrary intersections, and hence that $Q = F_{\cap Q}$. However, for completeness we give a proof of this. Let $Q = \{B_\alpha : \alpha < \kappa\}$. For $\alpha < \kappa$, define

$$C_\alpha = \bigcap_{\beta \leq \alpha} B_\beta$$

Also, let $C_\kappa = \bigcap_{\alpha<\kappa} B_\alpha$. Thus,

$$\beta < \alpha \leq \kappa \Longrightarrow C_\beta \supseteq C_\alpha$$

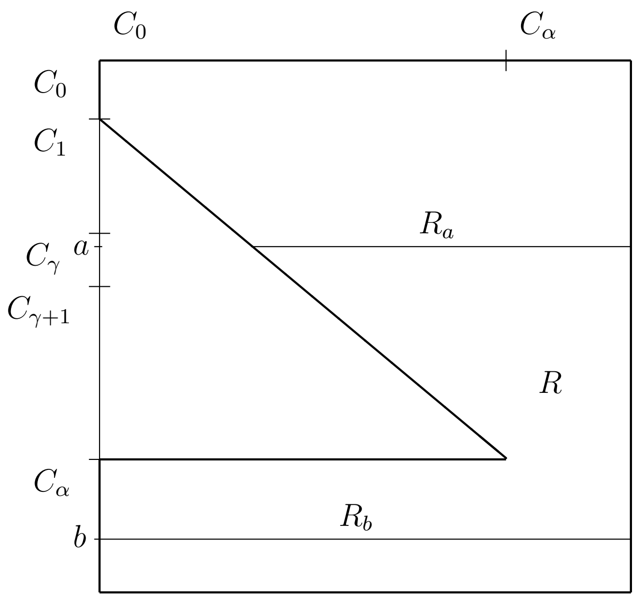
CLAIM: $C_\alpha \in Q$ for $\alpha \leq \kappa$.

The Claim is proved by induction on $\alpha$. Clearly it holds for $\alpha = 0$. For $\alpha = \beta + 1$ it follows from the induction hypothesis and closure under finite intersections. Let $\alpha$ be a limit ordinal, and suppose that $C_\alpha \notin Q$. Define $R$ by the following stipulations.

(i) If $a \in C_\alpha$, then $R_a = C_0 \in Q$.

(ii) If $a \in C_0 - C_\alpha$, then there is $\gamma + 1 < \alpha$ such that $a \in C_\gamma - C_{\gamma+1}$. Let $R_a = C_\gamma$. Thus $R_a \in Q$, by induction hypothesis.

$R$ may be pictured as the following subset of $C_0 \times C_0$:




It follows that $\{a : QR_a\} = C_0 \in Q$. Further, if $a \in C_\alpha$, then $(R^{-1})_a = C_0 \in Q$. And if $a \in C_0 - C_\alpha$, with $\gamma$ as in (ii) above,

$$(R^{-1})_a = C_\alpha \cup (C_0 - C_{\gamma+1}) = C'$$

If $C' \in Q$, it follows by closure under finite intersections that $C' \cap C_{\gamma+1} = C_\alpha \in Q$, contradicting our assumption. Thus, we have $C' \notin Q$. But then, $\{a : Q(R^{-1})_a\} = C_\alpha \notin Q$, contradicting self-commutativity. This proves the Claim.

*Case 2*: $Q$ is not closed under intersections.

By Lemma 4.2, $Q$ is splittable. Thus, by Lemma 4.1,

$$A \in Q \Longrightarrow \{a\} \in Q \text{ for some } a \in A \tag{6}$$

*Case 2A*: $Q$ is upward monotone.

Let $B = \{a : \{a\} \in Q\}$. Then, from monotonicity and (6),

$$A \in Q \Longleftrightarrow A \cap B \neq \emptyset$$

i.e., $Q = \bigcup_{a \in B} F_a$.

*Case 2B*: $Q$ is not upward monotone.

Again, let $B = \{a : \{a\} \in Q\}$. We first note that

$$a, b \in B, a \neq b \Longrightarrow \{a, b\} \notin Q \tag{7}$$

This follows, since otherwise we would have $\{a\} \in Q$, $\{a, b\} \in Q$, but also $\{a, b\} - \{a\} \in Q$, which contradicts Lemma 4.4.

Next, we make the

CLAIM: $Q$ is finite.

To see this, suppose $B$ is infinite. Take $a_0, a_1, a_2, \ldots \in B$ and define $R$ by

$$R_{a_n} = \{a_n, a_{n+1}\}, n = 0, 1, \ldots$$

By (7), $\{a : QR_a\} = \emptyset \notin Q$. But also, $\{a : Q(R^{-1})_a\} = \{a_0\} \in Q$, a contradiction, and the Claim is proved.

Finally, applying Lemmas 4.3 and 4.4 it is easy to prove by induction that

$$\forall B' \subseteq B : \; B' \in Q \Leftrightarrow |B'| \text{ is odd} \tag{8}$$

Thus, if $|A \cap B|$ is odd, $A \cap B \in Q$. But $A - (A \cap B) \notin Q$ by (6), since $a \in A - B$ implies $\{a\} \notin Q$. So by Lemma 4.3, $A \in Q$. Conversely, if $|A \cap B|$

is even, $A \cap B \notin Q$. Again, $A - (A \cap B) \notin Q$. Hence, by splittability, $A \notin Q$. We have shown that

$$A \in Q \iff |A \cap B| \text{ is odd},$$

i.e., that $Q = \bigoplus_{a \in B} F_a$. This concludes the proof. $\square$

# 5 Generalizations

We remarked earlier that the notion of self-commutativity has at least two natural generalizations to pairs $(Q_1, Q_2)$ of type $\langle 1 \rangle$ quantifiers. One is convertibility of the type $\langle 2 \rangle$ quantifier $Q_1 Q_2$. The other is the following notion of independence:

**Definition 5.1** $(Q_1, Q_2)$ is *independent* if, for all $R \subseteq M^2$,

$$Q_1 Q_2 R \Leftrightarrow Q_2 Q_1 R^{-1}$$

Thus, $Q$ is self-commuting iff $(Q, Q)$ is independent. I don't know the full answer to the following

**Problem:** Characterize the independent pairs.

In [3] it was shown that if $Q_1$ and $Q_2$ are upward monotone, non-trivial, and ISOM, $(Q_1, Q_2)$ is independent iff $Q_1 = Q_2 = \forall$ or $Q_1 = Q_2 = \exists$. But without these constraints there are lots of other independent pairs. By arguments similar to those used in establishing Facts 2.1 and 2.5 one verifies

**Fact 5.1** *If both $Q_1$ and $Q_2$ are unions of atoms, or both intersections of atoms, or both finite symmetric differences of atoms, then $(Q_1, Q_2)$ is independent.*

As to negations of symmetric differences, one can show using Fact 2.6 that the independent pairs are the following.

**Fact 5.2** *Let* $Q_1 = \bigoplus_{i=1}^n F_{a_i}$ *and* $Q_2 = \bigoplus_{j=1}^m F_{b_j}$, $n, m \geq 1$. *Then*

*a.* $(Q_1, \neg Q_2)$ *is independent iff* $n$ *is odd.*

*b.* $(\neg Q_1, Q_2)$ *is independent iff* $m$ *is odd.*

*c.* $(\neg Q_1, \neg Q_2)$ *is independent iff* $n, m$ *are both odd or both even.*

One might hope that Facts 5.1–2 give the only examples of independent pairs, but that is not so. The next proposition provides some further information.

Let us say that $Q$ *preserves unions of length* $\kappa$ if, for all $A_i \subseteq M, i \in I$ with $|I| \leq \kappa$,

$$Q(\bigcup_{i\in I} A_i) \Longleftrightarrow \bigvee_{i\in I} QA_i$$

Likewise, $Q$ *preserves intersections of length* $\kappa$ if

$$Q(\bigcap_{i\in I} A_i) \Longleftrightarrow \bigwedge_{i\in I} QA_i$$

Note first that if $Q$ preserves finite unions or intersections, then $Q$ is upward monotone. (In particular, $Q$ preserves finite intersections iff $Q$ is a filter.) Using this, it is not hard to verify that if $|M|, |Q| > 1$, then

(9) $\quad Q$ preserves unions of length $|M| \iff \exists B \subseteq M(Q = \bigcup_{a\in B} F_a)$

(10) $Q$ preserves intersections of length $|Q| \iff \exists B \subseteq M(Q = \bigcap_{a\in B} F_a)$

**Proposition 5.3** *If one of* $Q_1, Q_2$ *is* $\bigcup_{a\in B} F_a$ $(\bigcap_{a\in B} F_a)$, *then* $(Q_1, Q_2)$ *is independent iff the other quantifier preserves unions (intersections) of length* $|B|$.

*Proof*: Suppose $Q_1 = \bigcup_{a\in B} F_a$. We have

$$Q_1Q_2R \Longleftrightarrow \bigvee_{a\in B} Q_2R_a$$

$$Q_2Q_1R^{-1} \Longleftrightarrow Q_2(\bigcup_{a\in B} R_a)$$

So the 'if' direction is clear. For the other direction, suppose there are $A_i \subseteq M, i \in I$ with $|I| \leq |B|$ such that either $Q_2(\bigcup_{i\in I} A_i)$ and for all $i \in I$,

$\neg Q_2 A_i$, or $\neg Q_2(\bigcup_{i\in I} A_i)$ and for some $i \in I$, $Q_2 A_i$. Since $|I| \leq |B|$ we can define $R$ such that for all $i \in I$, $A_i = R_a$ for some $a \in B$, and for all $a \in B$, $R_a = A_i$ for some $i \in I$. Thus $\bigcup_{i\in I} A_i = \bigcup_{a\in B} R_a$, and it follows that $(Q_1, Q_2)$ is not independent. The case of intersection is similar. $\square$

Note the special case when $|B| = 1$, i.e., when one of $Q_1, Q_2$ is $F_a$. Then there is no constraint on the other quantifier, which is precisely to say that $F_a$ is *scope-independent*.

The proposition provides examples of independent pairs not covered by Fact 5.1. For example, if $Q_1$ is a principal filter $F_B$ with $B$ finite and $Q_2$ is *any* filter, $(Q_1, Q_2)$ is independent.

We end by looking at another generalization, namely, to type $\langle 1, 1\rangle$ quantifiers. These quantifiers are ubiquitous in natural language semantics, as denotations of *determiners* like *every, no, most, between three and six, all but seven, every ... except John, several of Mary's.* The first argument of such a quantifier is then called the *noun argument*, and the second the *verb argument*.

Fixing the noun argument of a determiner denotation gives a noun phrase denotation. Formally:

**Definition 5.2** If $Q$ is a type $\langle 1, 1\rangle$ quantifier on $M$ and $A \subseteq M$, the type $\langle 1\rangle$ quantifier $Q^A$ on $M$ is defined by

$$Q^A B \Leftrightarrow QAB$$

for all $B \subseteq M$.

Using this we can combine (iterate) two type $\langle 1, 1\rangle$ quantifiers by reduction to the type $\langle 1\rangle$ case (section 1 (2)).

**Definition 5.3** If $Q_1, Q_2$ are of type $\langle 1, 1\rangle$ we define the type $\langle 1, 1, 2\rangle$ quantifier $Q_1 Q_2$ by

$$Q_1 Q_2 ABR \Longleftrightarrow Q_1^A Q_2^B R$$

for $A, B \subseteq M$ and $R \subseteq M^2$.

$Q_1 Q_2 ABR$ interprets standard sentences with a transitive verb and quantified subject and object, like *Most critics reviewed five films* and *Every professor except John read Mary's book.*

We now say that a type $\langle 1,1\rangle$ quantifier $Q$ is *self-commuting* (on $M$) if, for all $A, B \subseteq M$ and $R \subseteq M^2$,

$$Q^A Q^B R \Longleftrightarrow Q^B Q^A R^{-1} \tag{11}$$

Thus, $Q$ is self-commuting iff each pair $(Q^A, Q^B)$ is independent. We can obtain a characterization of the self-commuting type $\langle 1,1\rangle$ quantifiers as a corollary to our theorem. To simplify the statement, let us say that $Q$ is *good* if whenever $Q^A$ and $Q^B$ are one of $\mathbf{0}, \mathbf{1}, \bigoplus_{i=1}^n F_{a_i}$, or $\neg \bigoplus_{i=1}^n F_{a_i}$, then $(Q^A, Q^B)$ is independent. The necessary and sufficient conditions for this are easily described explicitly, without reference to independence, using Facts 5.1–2.

**Corollary 5.4** *A type $\langle 1,1\rangle$ quantifier $Q$ is self-commuting iff $Q$ is good, and* either *each $Q^A$ is $\mathbf{1}$ or a union of atoms,* or *each $Q^A$ is $\mathbf{0}$ or an intersection of atoms,* or *each $Q^A$ is $\mathbf{0}$ or $\mathbf{1}$ or a finite symmetric difference of atoms or the negation of such a symmetric difference.*

*Proof*: The 'if' direction follows from Facts 5.1–2. For the other direction, we use the fact that if $Q$ is self-commuting then each $Q^A$ is self-commuting, hence by the theorem a union or intersection of atoms, or (the negation of) a finite symmetric difference of atoms. It now suffices to go through the various possible cases.

*Case 1*: $\exists A(Q^A = \bigcup_{a\in D} F_a)$, for some $D$ with $|D| > 1$.

Take any $B$. $(Q^A, Q^B)$ is independent, so by Proposition 5.3, $Q^B$ preserves finite unions; in particular it is upward monotone. Also, $Q^B$ is self-commuting. But it cannot be an intersection of at least two atoms, since such intersections do not preserve finite unions. It cannot be a finite symmetric difference of at least two atoms, since such symmetric differences are not upward monotone. For the same reason, it cannot be the negation of a finite symmetric difference of atoms. Thus, it has to be a union of atoms (possibly an empty union, or a single atom), or $\mathbf{1}$.

*Case 2*: $\exists A(Q^A = \bigcap_{a\in D} F_a)$, for some $D$ with $|D| > 1$.

By a similar argument, every $Q^B$ must then be an intersection of atoms, or $\mathbf{0}$.

*Case 3*: $\exists A(Q^A = \bigoplus_{i=1}^n F_{a_i})$, for some $n > 1$.

Consider again any $Q^B$. By the reasoning in Cases 1 and 2, $Q^B$ cannot be a union or intersection of at least two atoms. Hence, it must be $\mathbf{0}$ or $\mathbf{1}$ or a finite symmetric difference of atoms, or the negation of such a symmetric difference (provided $n$ is odd, by Fact 5.2).

*Case 4*: $\exists A(Q^A = \neg \bigoplus_{i=1}^{n} F_{a_i})$, for some $n \geq 1$.

Similar to Case 3.

*Case 5*: Not Case 1–4.

Since each $Q^A$ is self-commuting, the only possibility left is that every $Q^A$ is $\mathbf{0}$ or $\mathbf{1}$ or an atom. This finishes the proof. $\Box$

Returning to type $\langle 1, 1\rangle$ quantifiers as denotations of English determiners, there don't seem to be any whose corresponding noun phrase denotations are (negations of) symmetric differences. So disregarding these, the corollary says basically that if $Q$ is self-commuting then either for every $A$ there is $D$ such that $Q^A = \mathit{some}^D$ $(= \bigcup_{a\in D} F_a)$, or for every $A$ there is $D$ such that $Q^A = \mathit{all}^D$ $(= \bigcap_{a\in D} F_a)$.

Examples of English determiners denoting self-commuting quantifiers are *some, all, John's, the ten students'*. Of course, many other type $\langle 1, 1\rangle$ quantifiers fulfill the requirement, for example, a $Q$ such that $Q^A$ is *John's books* when $A$ is the set of books (so $D$ is the set of books that John owns), and $Q^A$ is *Mary's bikes* when $A$ is the set of bikes (then $D$ is the set of bikes that Mary owns).

Type $\langle 1, 1\rangle$ quantifiers denoted by determiners in natural languages obey certain constraints, most typically *conservativity*: for all $A, B \subseteq M$,

$$QAB \Leftrightarrow QA\, A\cap B$$

The effect of conservativity here is that $D \subseteq A$ in the condition above. This does not rule out the example with John's books and Mary's bikes, however. But if we also assume ISOM, one can show (cf. [4]) that it follows that $D = A$. Then, self-commutativity implies that $Q$ is either (the denotation of) *some* or *all* on $M$.

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